Chapter 6.

Convergence to the limit set of linear cellular automata, II

1 Introduction

A cellular automaton consists of a finite-dimensional lattice of sites, each of which takes an element of a finite set $\mathbb{Z}_{p^r} = \{0, 1, \ldots, p^r - 1\}$ of integers at each time step and the value of each site at any time step is determined as a function of the values of the neighbouring sites at the previous time step.

We introduce the set $\mathcal{P}$ of all configurations $a : \mathbb{Z}^d \to \mathbb{Z}_{p^r}$ with compact support (i.e., $\#\{i \mid a(i) \neq 0\} < \infty$) and define a linear rule L in $\mathcal{P}$ as

$$(La)(x) = \sum_{j=1}^{m} \alpha_j a(x + k_j) \qquad (\text{mod } p^r). \tag{1.1}$$

The configuration of cellular automata at time step t is represented by operating L on the initial configuration by t times.

In case of $p = 2$ and $r = 1$, S. J. Willson [6] investigated the so-called limit set of LCA. For $n \in \mathbb{Z}_+$ and $a \in \mathcal{P}$, he considered the set

$$K(n, a) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq 2^n,\ (L^t a)(x) = 1\},$$

where L^t is the t-th power of L. He showed that there exists the limit set of $K(n, a)/2^n$ for any nonzero $a \in \mathcal{P}$ in the sense of Kuratowski limit [1, 4] and that the limit set does not depend on an initial configuration. The limit set of LCA for a certain linear rule is a Sierpinski gasket-like pattern.

In case of mod p, the sets $\limsup K(n, \delta)/p^n$ and $\liminf K(n, \delta)/p^n$ in the sense of Kuratowski limit are the same with $\bigcap_{k=1}^{\infty} \overline{\bigcup_{n \geq k} K(n, \delta)/p^n}$ and $\overline{\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K(n, \delta)/p^n}$ respectively, since $\{K(n, \delta)/p^n\}$ is an increasing sequence. So in [3], we considered the limit set in the sense of the set theory, which is defined if the set $\bigcap_{k=1}^{\infty} \bigcup_{n \geq k} K(n, a)/p^n$ coincides with the set $\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} K(n, a)/p^n$ without taking its closure.

As an extension of the result of Willson, S. Takahashi [5] investigated the case of an arbitrary prime number $p \geq 2$ and $r \in \mathbb{N}$ and he considered the set

$$K(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ (L^t\delta)(x) \neq 0\}$$

for $n \in \mathbb{Z}_+$. By using the set $K(n, \delta)$, he also defined the limit set as a subset of $\mathbb{R}^d \times [0, 1]$ in the same way as the case of $p = 2$, and showed the existence of the limit set Y_δ of $\{K(n, \delta)/p^n\}$. Takahashi also investigated the limit set of "b-state" $K_b(n, \delta) = \{(x, t) \in \mathbb{Z}^d \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ (L^t\delta)(x) = b\}$ for $j \in \{1, \ldots, p^r - 1\}$.

By defining the metric D_f, we considered the convergence of $\mathbb{Z}_p$-valued functions $\psi_n(\delta)$ on $\mathbb{R}^d \times [0, 1]$ which corresponds to the values of sites up to the p^n-th time step of LCA and expresses all the states simultaneously in [3].

In this paper, we extend the result in [3] to the case of mod p^r, where p is prime and $r \in \mathbb{N}$. We show that there exists the limit function in the pointwise topology (Theorem 2.3). In Section 3, we define two metric d_f, D_f in the space USC of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0, 1]$ and give the result concerning d_f and D_f (Theorem 3.1). In Section 4, we investigate the convergence of $\{\psi_n(\delta)\}$ in these two metrics in the space USC with $\mathbb{R} \times [0, 1]$. We show that $\{\psi_n(\delta)\}$ is a Cauchy sequence in the metric d_f and $\psi_n(\delta)$ converges to the function f_δ in the metric D_f (Theorem 4.1) and that the similar results hold for any nonzero initial configuration $a \in \mathcal{P}$ (Theorem 4.14). In Section 5, we consider the relation between the limit function with respect to D_f and the limit set in the sense of Kuratowski limit. We show that the upper envelope of g_δ, which is the limit function of $\{\psi_n(\delta)\}$ in the pointwise topology, corresponds to the limit sets in the sense of Kuratowski limit and that f_δ is the upper envelope of g_δ (Theorem 5.2). For a nonzero configuration $a \in \mathcal{P}$, we show that the upper envelope of g_a, which is the limit function of $\{\psi_n(a)\}$ in the pointwise topology, corresponds to the limit sets in the sense of Kuratowski limit (Theorem 5.3) and this implies that the upper envelope of g_a depends on only the value $a(0)$. We prove the relation between the upper envelope of g_a and the limit function of $\{\psi_n(a)\}$ in the metric D_f and the limit function depends on all values

$a(x)$ $(x \in \mathbb{Z})$ (Theorem 5.4). This theorem implies that the upper envelope of g_a is not always equal to the limit function though both are the same in the case of mod p. While the limit function always takes two values in the case of mod p, it occurs the limit function takes more than three values in the case of mod p^r.

2 Convergence in the pointwise topology

We define a d-dimensional p^r-state *linear cellular automata* (LCA) as follows:

Let p be a prime number and let $\mathcal{P}$ be the set of all configurations $a : \mathbb{Z}^d \to \mathbb{Z}_{p^r}$ with compact support. We define $\delta \in \mathcal{P}$ as

$$\delta(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0. \end{cases}$$

Let L:$\mathcal{P} \to \mathcal{P}$ mod p^r be a linear transition rule as follows:

$$(La)(x) = \sum_{j \in G} \alpha_j a(x + k_j) \qquad \text{for } a \in \mathcal{P}, \tag{2.1}$$

where G is a finite subset of $\mathbb{Z}$ with $\sharp G \geq 2$, $k_j \in \mathbb{Z}^d$ $(j \in G)$ is a neighbouring site of origin, $\alpha_k \in \mathbb{Z}_{p^r} \backslash \{0\}$ and *the summation* $\sum$ *is taken as the summation with mod* p^r *throughout this paper.*

Let

$$X_n = \{(\frac{x}{p^n}, \frac{t}{p^n}) \in \mathbb{R}^d \times [0,1] \mid x \in \mathbb{Z}^d, t \in \mathbb{Z}_+, 0 \leq t \leq p^n\}$$

for $n \in \mathbb{Z}_+$ and put

$$G_j = \{\ell \in \mathbb{Z}^d \mid (L^j\delta)(\ell) \neq 0\} \tag{2.2}$$

for $j \in \mathbb{Z}_+$.

Define a map ψ_n from $\mathcal{P}$ to the function space on $\mathbb{R}^d \times [0,1]$ for $a \in \mathcal{P}$ and $n \in \mathbb{Z}_+$ by

$$(\psi_n(a))(\frac{x}{p^n}, \frac{t}{p^n}) = \begin{cases} (L^t a)(x) & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in X_n, \\ 0 & \text{if } (\frac{x}{p^n}, \frac{t}{p^n}) \in (\mathbb{R}^d \times [0,1]) \setminus X_n \end{cases} \tag{2.3}$$

and a map $S_{\ell,j}$: $\mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [\frac{j}{p}, \frac{j+1}{p}]$ by

$$S_{\ell,j}(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{\ell}{p^r}, \frac{j}{p}). \tag{2.4}$$

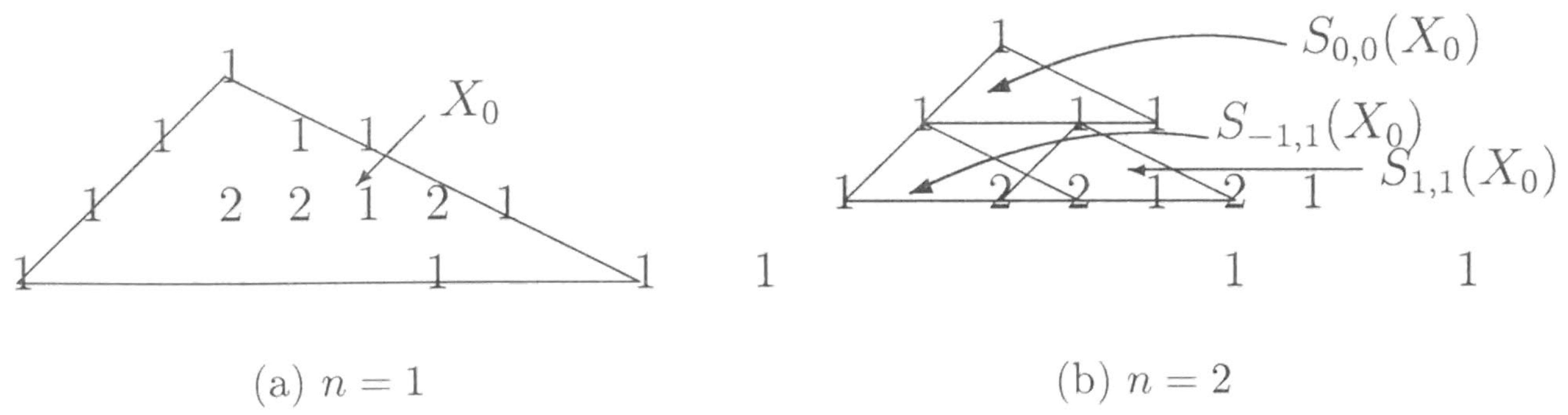


Figure 1: An exapmle of maps $S_{\ell,j}$ with $La(x) = a(x-2) + a(x-1) + a(x+1) \pmod 3$.

For a function g on $\mathbb{R}^d \times [0,1]$, by using maps $S_{\ell,j}$ define a function Tg on $\mathbb{R}^d \times [0,1]$ by

$$Tg(y,q) = \sum_{\ell \in G_{jp^{r-1}}} (L^{jp^{r-1}}\delta)(\ell) g(S_{\ell,j}^{-1}(y,q)) \tag{2.5}$$

for $\frac{j}{p} < q \le \frac{j+1}{p}$ with $0 \le j \le p-1$ and

$$Tg(y,0) = g(py,0).$$

Lemma 2.1. [5] *Let L be a linear cellular automata defined as (2.1) with* mod p^r. *Then for $j,l \in \mathbb{Z}_+$, we have*

$$L^{p^{l+r-1}j}\delta(x) = \begin{cases} L^{jp^{r-1}}\delta(y) & \text{if there exists } y \text{ such that } p^l y = x, \\ 0 & \text{otherwise.} \end{cases}$$

We have Lemma 2.2 in a similar way to the case of mod p [3, Lemma 2.3].

Lemma 2.2. *For $a \in \mathcal{P}$ and $j,n,i \in \mathbb{Z}_+$, we have*

$$(L^{jp^{n+r-1}+i}a)(x) = \sum_{\ell \in G_{jp^{r-1}}} (L^{jp^{r-1}}\delta)(\ell)(L^i a)(x - \ell p^{n+r-1}) \tag{2.6}$$

Using the above lemmas, we can show the following theorem in a similar way to the case of mod p [3, Theorem 2.5].

Theorem 2.3. *For $a \in \mathcal{P}$ with $a(0) \ne 0$, we have the following assertions:*

(1) *The sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R}^d \times [0,1]$ in the pointwise topology.*

(2) *The limit function g_a of the sequence $\{\psi_n(a)\}$ in the pointwise topology is T-invariant, that is, $Tg_a = g_a$.*

(3) *As for the limit functions g_δ and g_a of $\{\psi_n(\delta)\}$ and $\{\psi_n(a)\}$ respectively, we have $a(0)g_\delta = g_a$.*

Proof. (1) For $n \in \mathbb{Z}_+$ satisfying $n > r-1$, let $X'_n = \{(\frac{x}{p^n}, \frac{jp^{r-1}}{p^n}) \mid x \in \mathbb{Z}^d, j = 0, 1, \ldots, p^{n-r+1}\}$. Then we have $\cup_{n=1}^{\infty} X'_{n+r-1} = \cup_{n=1}^{\infty} X_n$. For $(y,q) \in \mathbb{R}^d \times [0,1] \setminus \cup_{n=1}^{\infty} X_{n+r-1}$,

$$(\psi_n(a))(y,q) = 0$$

by the definition of ψ_n.

For $(y,q) \in \cup_{n=1}^{\infty} X'_{n+r-1}$, we show there exists $\lim_{n\to\infty}(\psi_n(a))(y,q)$ in the same way as the case of mod p. So the sequence $\{\psi_n(a)\}$ converges to a function on $\mathbb{R} \times [0,1]$ in the pointwise topology.

(2) and (3) are proved in the same way as the case of mod p. $\square$

3 The space of $\mathbb{Z}_{p^r}$-valued upper semi continuous functions

In this section, we shall introduce two metrics d_f, D_f in the space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on a compact subset of $\mathbb{R}^d \times [0,1]$. Let USC be the space of $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions on $\mathbb{R}^d \times [0,1]$, where $\mathbb{Z}_{p^r}$-valued upper semi-continuous functions mean upper semi-continuous functions embedded in $\mathbb{R}$-valued function spaces. For functions f, $g \in USC$, the order $f \geq g$ is defined by $f(y,q) \geq g(y,q)$ for any $(y,q) \in \mathbb{R}^d \times [0,1]$ by considering $\mathbb{Z}_{p^r}$ as a subset of $\mathbb{R}$. For functions $\{f_\lambda\}_{\lambda\in\Lambda} \subset USC$ having an upper bound, let

$$g_1(y,q) = \inf\{g(y,q) \mid g \in USC, g \geq f_\lambda \text{ for any } \lambda \in \Lambda\}$$

and

$$g_2(y,q) = \inf\{f_\lambda(y,q) \mid \lambda \in \Lambda\}.$$

Then g_1 and g_2 belong to USC and g_1 is the least upper bound function $\bigvee f_\lambda$ and g_2 is the greatest lower bound function $\bigwedge f_\lambda$ in USC. So the space USC is an order complete lattice.

Let K be a compact subset of $\mathbb{R}^d \times [0,1]$ and (y_0, q_0) be a point of $(\mathbb{R}^d \times [0,1]) \setminus K$. Let

$$USC|_K = \{g \in USC \mid \text{ support of } g \subset K\}.$$

By using the Hausdorff distance $D(A,B)$ of non-empty compact sets A and B in $\mathbb{R}^d \times [0,1]$, we shall define the pseudodistance $D_0(A,B)$ of A and B in $\mathbb{R}^d \times [0,1]$ by

$$D_0(A,B) = D(A \cup \{(y_0, q_0)\}, B \cup \{(y_0, q_0)\})$$

and metrics d_f, D_f in $USC|_K$ as follows:

$$d_f(g_1, g_2) = \max\nolimits_{1 \le j \le p^r - 1} D_0(\overline{g_1^{-1}(j)}, \overline{g_2^{-1}(j)}),$$

$$D_f(g_1, g_2) = \max\nolimits_{1 \le s \le p^r - 1} D_0(g_1^{-1}[s+], g_2^{-1}[s+])$$

for $g_1, g_2 \in USC|_K$, where $g^{-1}[s+] = \{(x,t) \mid g(x,t) \ge s\}$ and $\overline{g_1^{-1}(j)}$ is the closure of the set $g_1^{-1}(j) = \{(x,t) \mid g(x,t) = j\}$. It is easy to see that d_f and D_f satisfy the axioms of metric in $USC|_K$. Then we can show the following theorem in a similar way to Theorem 3.5 in [3].

Theorem 3.1. *For $\{f_n\} \subset USC|_K$, suppose $d_f(f_n, f_m) \to 0$ as $n, m \to \infty$. Let $g = \bigwedge_{k=1}^{\infty} \bigvee_{n \ge k} f_n$. Then we have*

$$D_f(f_n, g) \to 0 \text{ as } n \to \infty.$$

Using the metrics d_f and D_f, we consider the convergence to the limit set.

4 Convergence of $\psi_n(\delta)$ in case of $\mathbb{R} \times [0,1]$

In this section, we will consider $\mathbb{Z}_{p^r}$-valued upper semi continuous functions on $\mathbb{R} \times [0,1]$ and show $\psi_n(\delta)$ converges to the limit function with respect to the metric D_f. We first introduce some notation.

Let α_k be defined in (2.1) and and suppose $k_i < k_j (i < j)$ for $i, j \in G$, which is defined in (2.1). Put

$$k_- = \min\{j \mid \alpha_j \neq 0 \text{ for } j \in G\},$$

$$k_+ = \max\{j \mid \alpha_j \neq 0 \text{ for } j \in G\}$$

and

$$k_0 = k_+ - k_-. \tag{4.1}$$

For $j \in \{0, 1, \ldots, p\}$, put

$$r_j = j + \frac{j(j-1)p^{r-1}}{2} k_0.$$

For convenience, we define a map $S_\ell \colon \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [\frac{j}{p}, \frac{j+1}{p}]$, which has the correspondence with some $S_{\ell,j}$ of (2.4), by

$$S_\ell(x,t) = (\frac{x}{p}, \frac{t}{p}) + (\frac{-jp^{r-1}k_+ + i - 1}{p^r}, \frac{j}{p}) \tag{4.2}$$

with $\ell = r_j + i (j \in \{0, 1, \ldots, p\}, i \in \{1, 2, \ldots, jp^{r-1}k_0 + 1\})$ and put

$$c_\ell = L^{jp^{r-1}} \delta(-jp^{r-1}k_+ + i - 1)$$

and

$$\Lambda = \{\ell \in \{1, \ldots, r_p\} \mid c_\ell \neq 0\}.$$

Then for $(y, q) \in \mathbb{R} \times [0,1]$ satisfying $\frac{j}{p} \leq q \leq \frac{j+1}{p}$ with $0 \leq j \leq p-1$, we have

$$(\psi_{n+1}(\delta))(y,q) = \sum_{\ell = r_j + 1}^{r_{j+1}} c_\ell (\psi_n(\delta))(S_\ell^{-1}(y,q)). \tag{4.3}$$

Let X_0 be the smallest convex subset of $\mathbb{R} \times [0,1]$ containing the support of $\psi_1(\delta)$, that is,

$$X_0 = \{(y,q) \in \mathbb{R} \times [0,1] \mid 0 \leq q \leq 1, -qk_+ \leq y \leq -qk_-\}. \tag{4.4}$$

Then for any $n \in \mathbb{Z}_+$, the support of $\psi_n(\delta)$ is contained in X_0 and for $\ell \in \Lambda$, $S_\ell(X_0)$ is also contained in X_0. So we consider the space $USC|_{X_0}$ and the metrics d_f, D_f in $USC|_{X_0}$ as in Section 3.

An element $j \in G$, which is defined in (2.1), is *prime* if $\alpha_j/p \notin \mathbb{Z}_+$. In this section, we shall show the following theorem.

Theorem 4.1. *Let the set G in (2.1) with mod p^r have at least two prime elements. Then we have*

(1) $d_f(\psi_n(\delta), \psi_m(\delta)) \to 0$ *as* $n, m \to \infty$.

(2) *Put* $f_0 = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_{n+r-1}(\delta)$, *where* $\bigwedge$ *and* $\bigvee$ *are lattice operations in* USC. *Then we have*

$$D_f(\psi_n(\delta), f_0) \to 0 \qquad as \qquad n \to \infty.$$

The way of the proof is a similar to that in the case of mod p [3, Theorem 4.1] as shown in the following.

4.1 Idea of the proof of Theorem 4.1

In case of mod p, we proved the lemmas and propositions in [3] by using the property that

$$(\psi_n(\delta))(y, q) = (\psi_{n+1}(\delta))(y, q) \qquad \text{for } (y, q) \in X_n$$

holds for any $n \in \mathbb{Z}_+$. In case of mod p^r, the equation above does not hold. Therefore we define a function H_n as follows. For $n \in \{r, r+1, r+2, \ldots\}$ let $X'_n = \{(\frac{x}{p^n}, \frac{jp^{r-1}}{p^n}) \mid x \in \mathbb{Z}, j = 0, 1, \ldots, p^{n-r+1}\}$ and

$$H_n = \psi_n(\delta) 1_{X'_n} \qquad \text{(see Figure 2)}. \tag{4.5}$$

By Lemma 2.1, we have

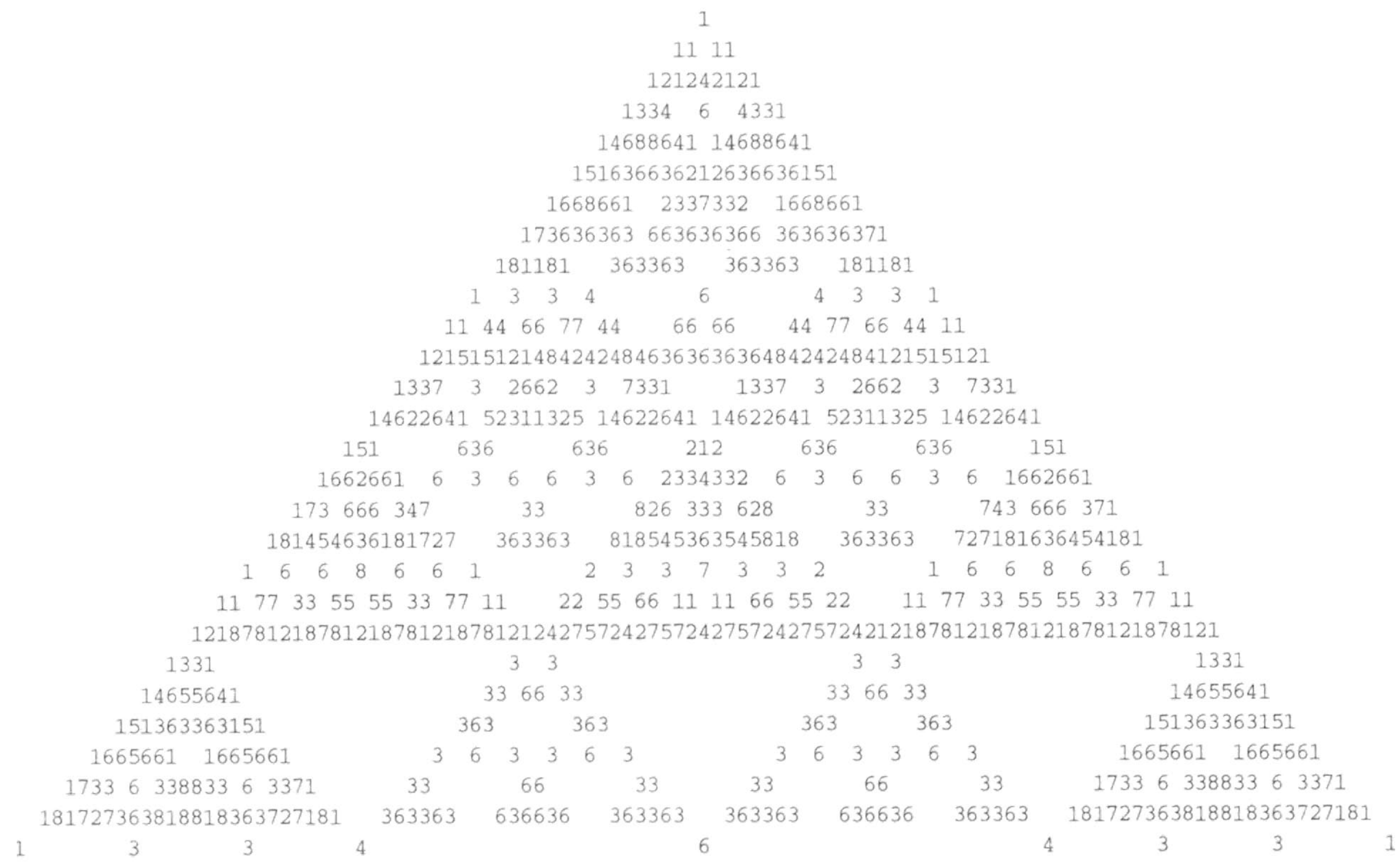
$$H_n(y, q) = H_{n+1}(y, q) \qquad \text{for } (y, q) \in X'_n.$$

In Section 4.2, we shall show

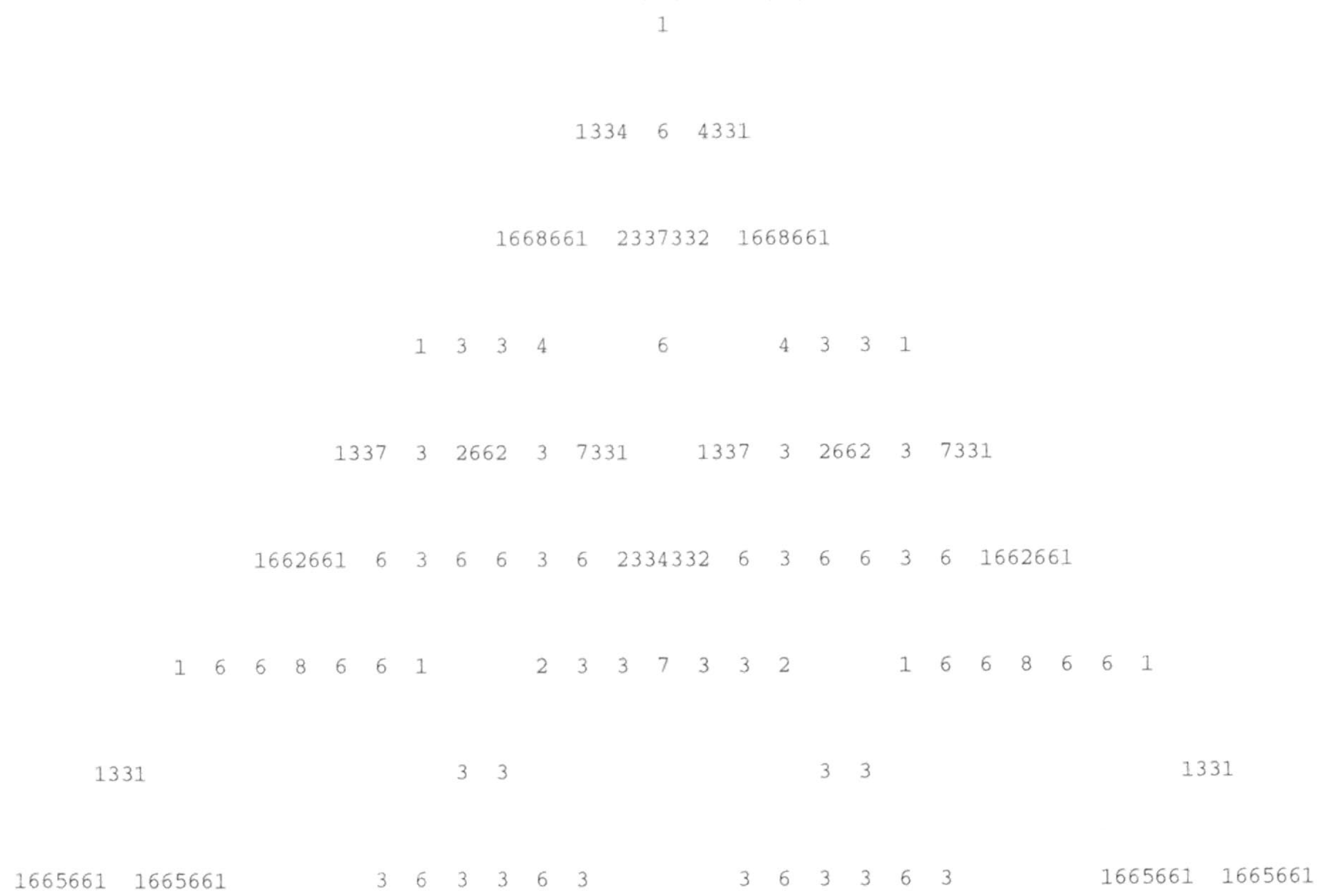
$$d_f(\psi_n(\delta), H_n) \to 0 \tag{4.6}$$

as $n \to \infty$. Then we shall only show the estimate

$$d_f(H_{n+1}, H_{m+1}) \leq \frac{1}{p} d_f(H_n, H_m). \tag{4.7}$$



(a) $\psi_3(\delta)$



(b) H_3

Figure 2: An example $\psi_3(\delta)$ and H_3 for $(La)(x) = a(x-2) + a(x-1) + a(x+1) + a(x+2) \pmod{3^2}$.

The inequality (4.7) can easily be verified if $\{(S_\ell(X_0))^\circ\}_{\ell\in\Lambda}$ is mutually disjoint, where $(S_\ell(X_0))^\circ$ is the interior of $S_\ell(X_0)$. However, the equation (4.7) is not easily obtained if $\{(S_\ell(X_0))^\circ\}_{\ell\in\Lambda}$ are mutually overlapped. Just as in the case of mod p, we introduce an auxiliary quantity $M_0^{n,n'}$ and show the following estimates:

M-1) $d_f(H_{n+1}, H_{n'+1}) \leq \frac{1}{p} M_0^{n,n'}$ (Proposition 4.11);

M-2) $M_0^{n+1,n'+1} \leq \frac{1}{p} M_0^{n,n'}$ (Proposition 4.12).

In order to define $M_0^{n,n'}$, we use two divisions $\{E_\gamma\}$ and $\{A_{b,j,s}\}$ of X_0 and functions $\{h_v^n\}$.

4.2 Relation between H_n and $\psi_n(\delta)$

We shall prove the following proposition in this section.

Proposition 4.2. *For the pseudodistance D_0 on $\mathbb{R}\times[0,1]$ and $\psi_n \in USC|_{X_0}$, we have*

$$D_0(H_n^{-1}[s+], (\psi_n(\delta))^{-1}[s+]) \to 0 \text{ as } n\to\infty \text{ for } s\in\{1,\ldots,p^r-1\}$$

and

$$D_0(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}) \to 0 \text{ as } n\to\infty \text{ for } j\in\{1,\ldots,p^r-1\}.$$

In order to show Proposition 4.2, we need the following

Proposition 4.3. *For a prime number p and $r\in\mathbb{N}$, let L be defined as (2.1) and the set G has at least two prime elements. Put $t(r,j) = j(p^r - p^{r-1})$ and $i(r,j) = -(t(r,j) - p^{r-1})k_{j_1} - p^{r-1}k_{j_2}$, where j_1 is maximum prime in G ,j_2 is the maximum prime element next to j_1 in G and k_j is defined in (2.1).*

When j ranges from 1 to p^r, $L^{t(r,j)}\delta(i(r,j))$ ranges from 0 to p^r-1.

Proof. By using the following Lemma 4.4, we can prove in a similar way to that of Theorem 2.7 in [2]. □

Lemma 4.4. *Suppose $r \geq 2$ and the set G has at least two prime elements. Then*

$$L^{t(r,j)}\delta(i(r,j)) \equiv L^{t(r,m)}\delta(i(r,m)) + sL^{t(r,p^{r-1})}\delta(i(r,p^{r-1})) \pmod{p^r}$$

holds for $j = sp^{r-1} + m$ with $s \in \{0, 1, \ldots, p-1\}$ and $m \in \{1, 2, \ldots, p^{r-1}\}$.

We have already proved a similar result to Lemma 4.4 and Proposition 4.3 when we supposed condition (A) in [2]. In this paper, we suppose that the set G has at least two prime elements instead of condition (A).

In order to verify Lemma 4.4, we need Lemma 4.5, 4.6 and 4.7.

Lemma 4.5 ([2], Lemma 2.2). *Suppose $q \in \mathbb{N}$ with $q/p \notin \mathbb{N}$, $t = jp^{r-1}$ with $j \in \mathbb{N}$ $v = p^l q$ with $l \in \{0, 1, \ldots, r-2\}$ and $v < t$.*

Then there exists $q' \in \mathbb{N}$ with $q'/p \notin \mathbb{N}$ such that $t - v = p^l q'$.

Lemma 4.6. *Put ${}_{a+b}\mathrm{C}_a = (a+b)!/(a!\,b!)$. Then*

$${}_{p^r - p^{r-1}}\mathrm{C}_i p^i \equiv 0 \pmod{p^r}$$

for $i \in \{1, 2, \ldots, p^r - p^{r-1}\}$.

Proof. Suppose $i = qp^\ell$ with $q \in \{1, 2, \ldots, p-1\}$ and $\ell \in \{0, 1, 2, \ldots, r-1\}$. There exists $b \in \mathbb{N}$ such that $b/p \notin \mathbb{N}$ and ${}_{p^r - p^{r-1}}\mathrm{C}_i p^i = bp^{r-1-\ell}p^{qp^\ell}$ by Lemma 4.5. Since $r - 1 - \ell + qp^\ell \geqq r$ holds, we obtain the conclusion. □

Put $m_0 = \sharp G$. In order to show the following lemmas, we first note that the value $(L^t\delta)(x)$ is expressed by

$$(L^t\delta)(x) = \sum \frac{t!}{u_1! \cdots u_{m_0}!} \alpha_{i_1}^{u_1} \cdots \alpha_{i_{m_0}}^{u_{m_0}} \pmod{p^r}, \tag{4.8}$$

where the summation is taken over $(u_1, \ldots, u_{m_0})$ such that $u_1 + \cdots + u_{m_0} = t$ and $-k_{i_1}u_1 - \cdots - k_{i_{m_0}}u_{m_0} = -x$. We also recall the relation

$$\frac{t!}{u_1! \cdots u_{m_0}!} = {}_t\mathrm{C}_{u_1} \times {}_{t-u_1}\mathrm{C}_{u_2} \times \cdots \times {}_{t-\sum_{i=1}^{m_0-1}}\mathrm{C}_{u_{m_0}}$$

and an element $j \in G$ is prime if $\alpha_j/p \notin \mathbb{N}$.

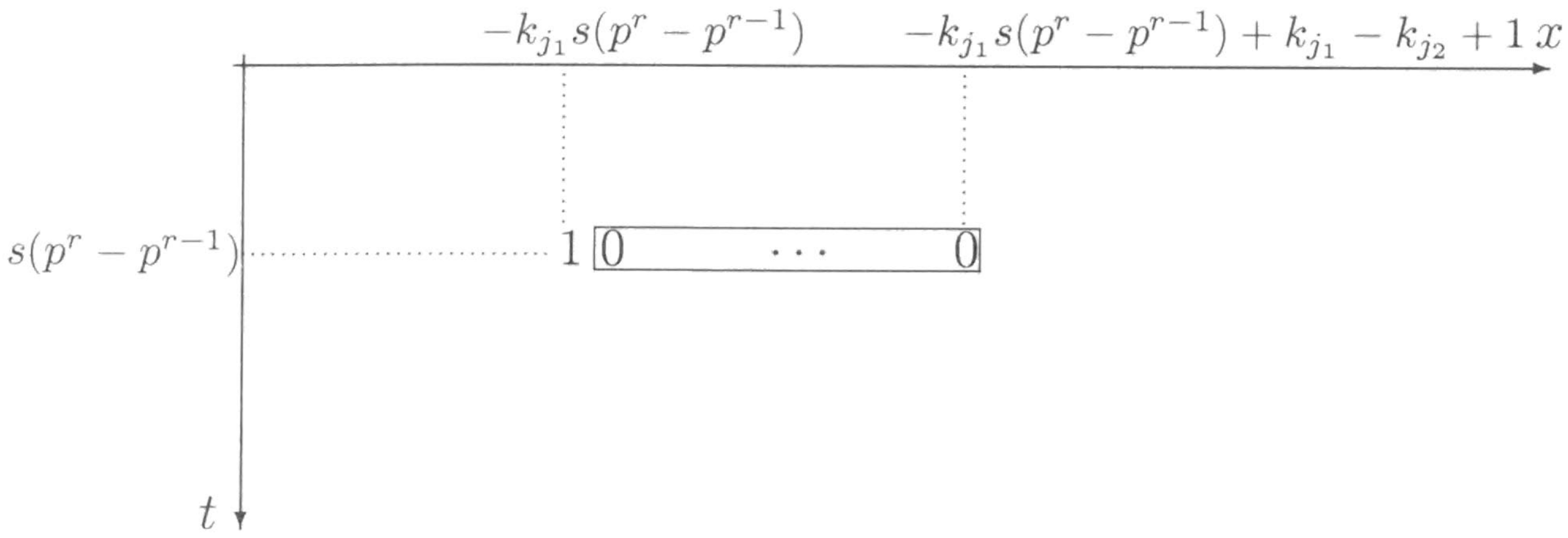


Figure 3: The values in the squared region is all 0.

Lemma 4.7. *Let the set G in (2.1) have at least two prime elements. Suppose that j_1 is maximum prime in G and that j_2 is the maximum prime element next to j_1 in G. Then*

$$L^{s(p^r-p^{r-1})}\delta(-k_{j_1}s(p^r - p^{r-1}) + \ell) \equiv 0 \pmod{p^r} \tag{4.9}$$

for $s \in \{1, 2, \ldots, p-1\}$ and $\ell \in \{1, 2, \ldots, k_{j_1} - k_{j_2} - 1\}$ (see Figure 3).

Proof. We note k_{j_1} is not expressed as a convex linear combination of other k_j, where $j \in G$ is prime. If there exists the path from $-k_{j_1}s(p^r - p^{r-1}) + \ell$ to the origin with $s(p^r - p^{r-1})$ time steps, then there exist $n_0 \in \mathbb{Z}_+$, $\{i_j \in \mathbb{N}\}_{j=1}^{n_0}$ and $\{m_j \in G\}_{j=1}^{n_0}$ such that

$$\sum_{j=1}^{n_0} i_j = s(p^r - p^{r-1}) \tag{4.10}$$

and

$$-k_{j_1}s(p^r - p^{r-1}) + \ell = -\sum_{j=1}^{n_0} i_j k_{m_j}. \tag{4.11}$$

Suppose m_j is prime for all $j \in \{1, \ldots, n_0\}$. From (4.10) and (4.11), $\ell = \sum_{j=1}^{n_0} i_j(k_{j_1} - k_{m_j})$ holds and there exists $j' \in \{1, \ldots, n_0\}$ such that $k_{m_{j'}} \leq k_{j_2}$. By $i_j \geq 1$, we obtain $\ell \geq k_{j_1} - k_{j_2}$, which contradicts $\ell \in \{1, 2, \ldots, k_{j_1} - k_{j_2} - 1\}$.

Therefore there exists $j \in \{1, 2, \ldots, n_0\}$ such that m_j is not prime. The equation (4.9) holds by Lemma 4.6 and (4.8). $\square$

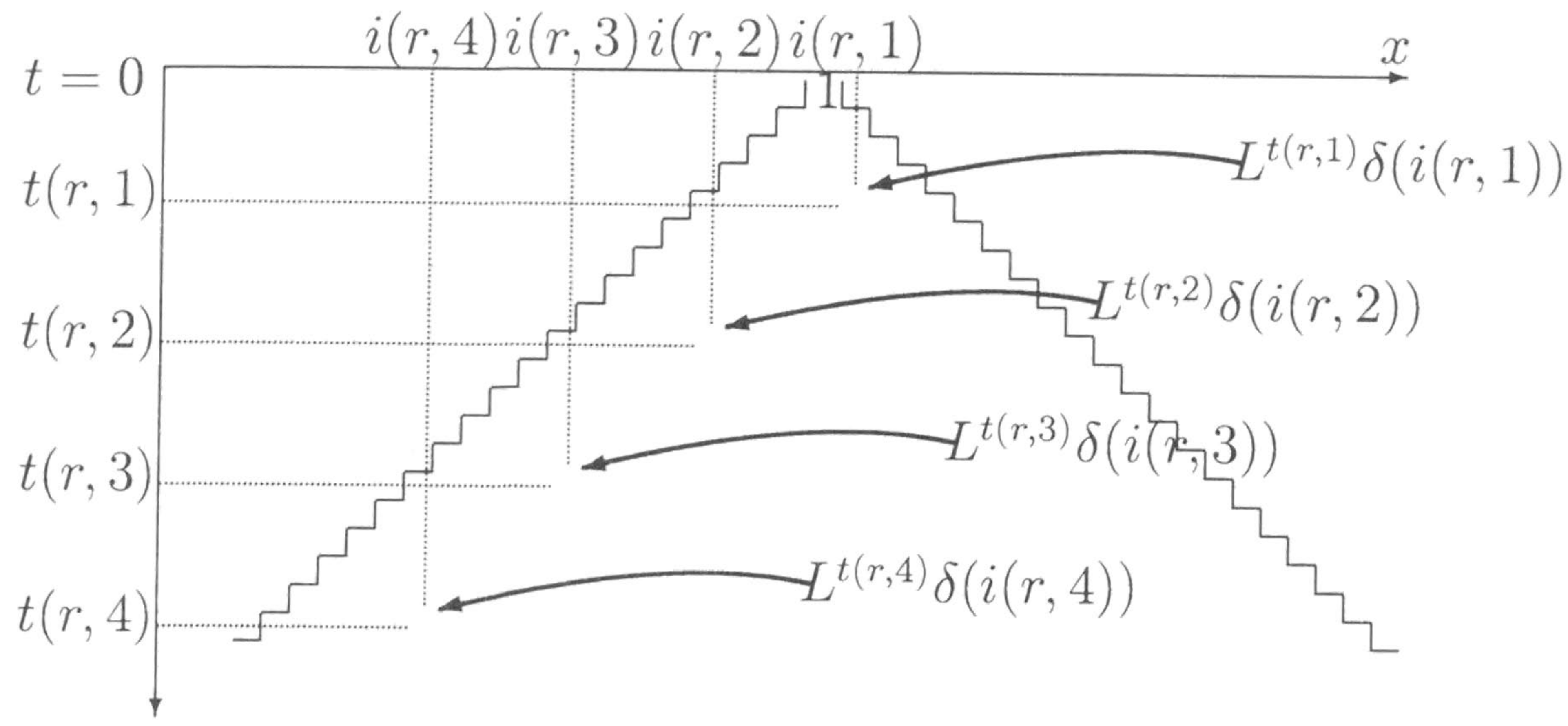


Figure 4: The relation among $t(r,j), i(r,j)$ and $L^{t(r,j)}\delta(i(r,j))$.

When the set G in (2.1) has at least two prime elements, suppose that j_1 is maximum prime in G and that j_2 is the maximum prime element next to j_1 in G. Using k_{j_1} and k_{j_2}, put

$$t(r,j) = j(p^r - p^{r-1}) \tag{4.12}$$

and

$$i(r,j) = -(t(r,j) - p^{r-1})k_{j_1} - p^{r-1}k_{j_2} \tag{4.13}$$

for $j \in \mathbb{N}$(see Figure 4).

Proof of Lemma 4.4.
When we compute $L^{t(r,j)}\delta(i(r,j))$ from the values at time $t(r,j-1)$, we need the values $L^{t(r,j-1)}\delta(x)$ with $x \in \{i(r,j) + k_-(p^r - p^{r-1}), \ldots, i(r,j) + k_+(p^r - p^{r-1})\}$ (see Figure 5). We note that the value $L^{t(r,j-1)}\delta(x)$ with $x \in \{i(r,j) + k_-(p^r - p^{r-1}), \ldots, -k_{j_1}t(r,j-1) - 1\}$ is a multiple of p by (4.8) and that the path from $i(r,j-1) + \ell$ to $i(r,j)$ for any $\ell \in \{1, \ldots, k_+(p^r - p^{r-1})\}$ with $p^r - p^{r-1}$ time steps needs at least one k_j, where $j \in G$ is not prime. Therefore by Lemma 4.6, the values $L^{t(r,j-1)}\delta(x)$ with $x \in \{i(r,j) + k_-(p^r - p^{r-1}), \ldots, -k_{j_1}t(r,j-1) - 1\} \cup \{i(r,j-1) + 1, \ldots, i(r,j) + k_+(p^r - p^{r-1})\}$ do not effect $L^{t(r,j)}\delta(i(r,j))$. So we

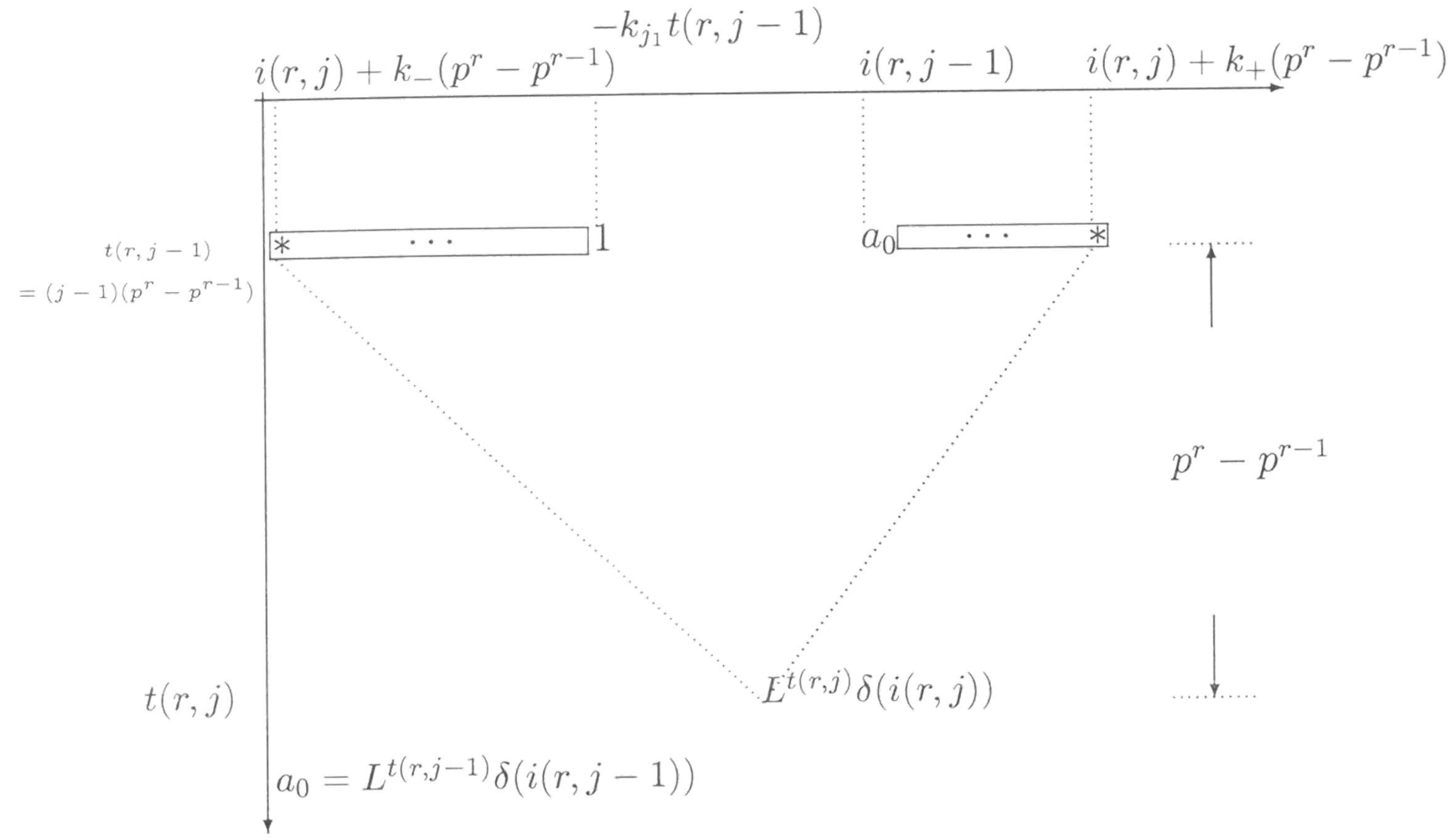


Figure 5: A region which effects the value $L^{t(r,j)}\delta(i(r,j))$. However we can ignore the squared regions by Lemma 4.6.

have

$$\begin{aligned} L^{t(r,j)}\delta(i(r,j)) \equiv {}_{p^r-p^{r-1}}\mathrm{C}_{p^{r-1}}\alpha_{j_1}^{p^r-2p^{r-1}}\alpha_{j_2}^{p^{r-1}} \\ + \sum_{\ell=1}^{p^{r-1}(k_{j_1}-k_{j_2})-1} B(r,\ell)b(r,j-1,\ell) \\ + \alpha_{j_1}^{p^r-p^{r-1}}L^{t(r,j-1)}\delta(i(r,j-1)) \pmod{p^r}, \quad (4.14) \end{aligned}$$

where $b(r,j,\ell) = L^{t(r,j)}\delta(-t(r,j)k_{j_1}+\ell)$ and $B(r,\ell)\in\mathbb{N}$. $B(r,\ell)$ is the number of the path from $-t(r,j)k_{j_1}+\ell$ to $i(r,j)$ with p^r-p^{r-1} time steps, and the number of the path from $-t(r,j)k_{j_1}+\ell$ to $i(r,j)$ with p^r-p^{r-1} time steps is the same as that from $-t(r,j')k_{j_1}+\ell$ to $i(r,j')$ with p^r-p^{r-1} time steps for any $j,j'\in\mathbb{Z}$. So $B(r,\ell)$ does not depend on j.

Using (4.14) and Lemma 4.7, we can show the conclusion in the same way as the case that L satisfies the condition (A). □

Proof of Proposition 4.2.

Suppose $n\in\{r,r+1,r+2,\ldots\}$. We have

$$D_0(H_n^{-1}[s+],(\psi_n(\delta))^{-1}[s+]) \le \max_{s\le j\le p^r-1} D_0(\overline{H_n^{-1}(j)},\overline{(\psi_n(\delta))^{-1}(j)}).$$

So we prove $D_0(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}) \to 0$ for all $j \in \{1, 2, \ldots, p^r - 1\}$ as $n \to \infty$.

Put

$$\begin{aligned} D_{0,r}(A, B) &= \sup\{d(A \cup \{(y_0, q_0)\}, y) \mid y \in B \cup \{(y_0, q_0)\}\}, \\ D_{0,\ell}(A, B) &= \sup\{d(x, B \cup \{(y_0, q_0)\}) \mid x \in A \cup \{(y_0, q_0)\}\} \end{aligned}$$

for compact sets A and B. Then we have

$$\begin{aligned} &D_0(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}) = \\ &\qquad \max\{D_{0,\ell}(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}), D_{0,r}(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)})\}. \end{aligned}$$

and

$$D_{0,\ell}(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}) = 0$$

for all n by the definition of H_n. So we will show for any $\epsilon > 0$ there exists $N \in \mathbb{Z}_+$ such that

$$D_{0,r}(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}) < \epsilon$$

for $n > N$.

Put

$$\begin{aligned} &t_0 = p^r(p^r - p^{r-1}) + 1, \\ &\ell_0 = \min\{\ell \in \mathbb{Z}_+ \mid t_0(k_+ - k_-) < p^\ell - 1\} \text{ (see Figure 6 (a))} \end{aligned}$$

and

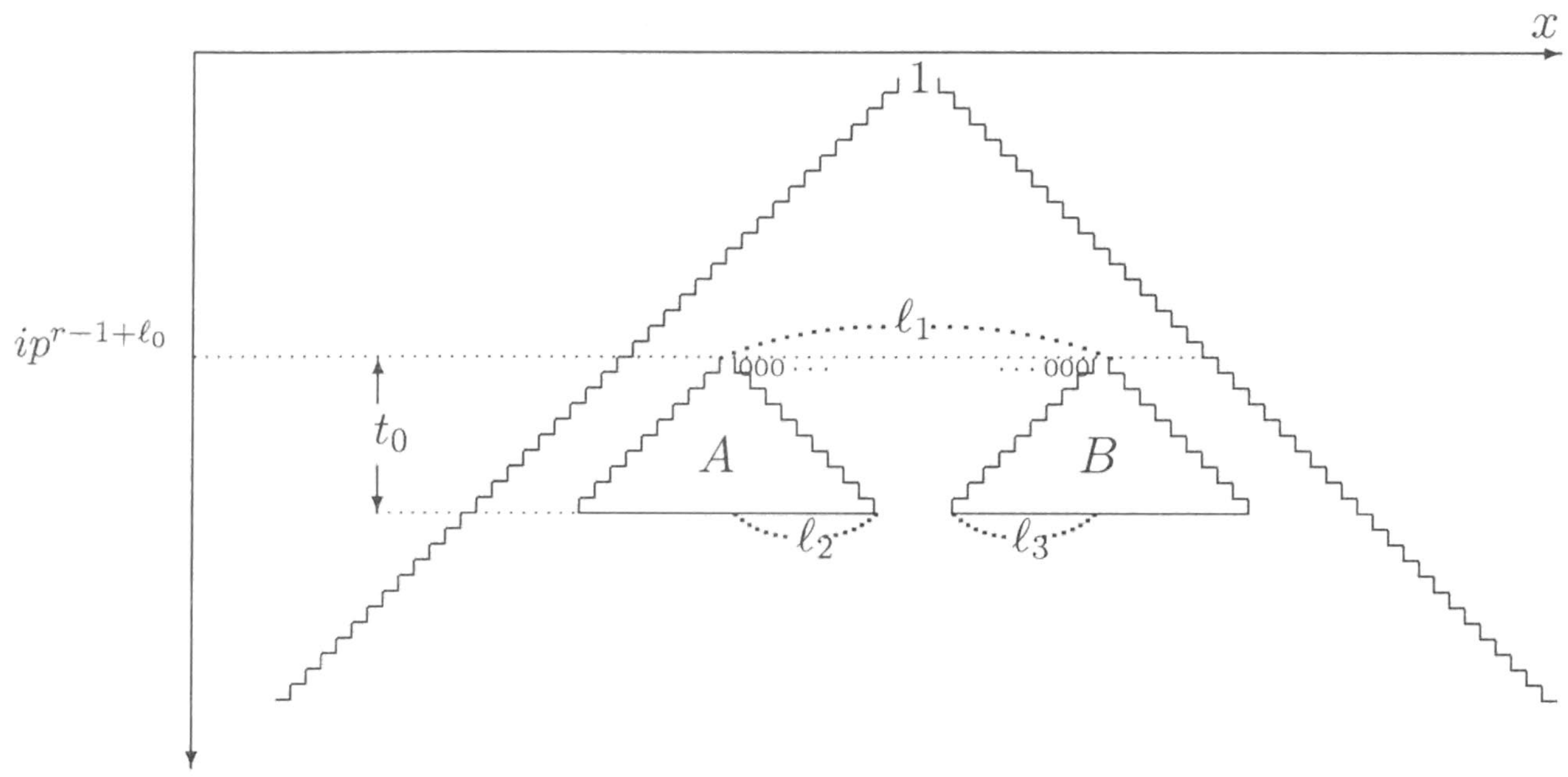
$$K(i) = \{x \in \mathbb{Z} \mid L^{ip^{r-1+\ell_0}}\delta(x) \neq 0\} \text{ for } i \in \mathbb{Z}_+.$$

For $n > r - 1 + \ell_0$ and $\boldsymbol{x} = (x/p^n, t/p^n) \in X_n \cap X_0$, put

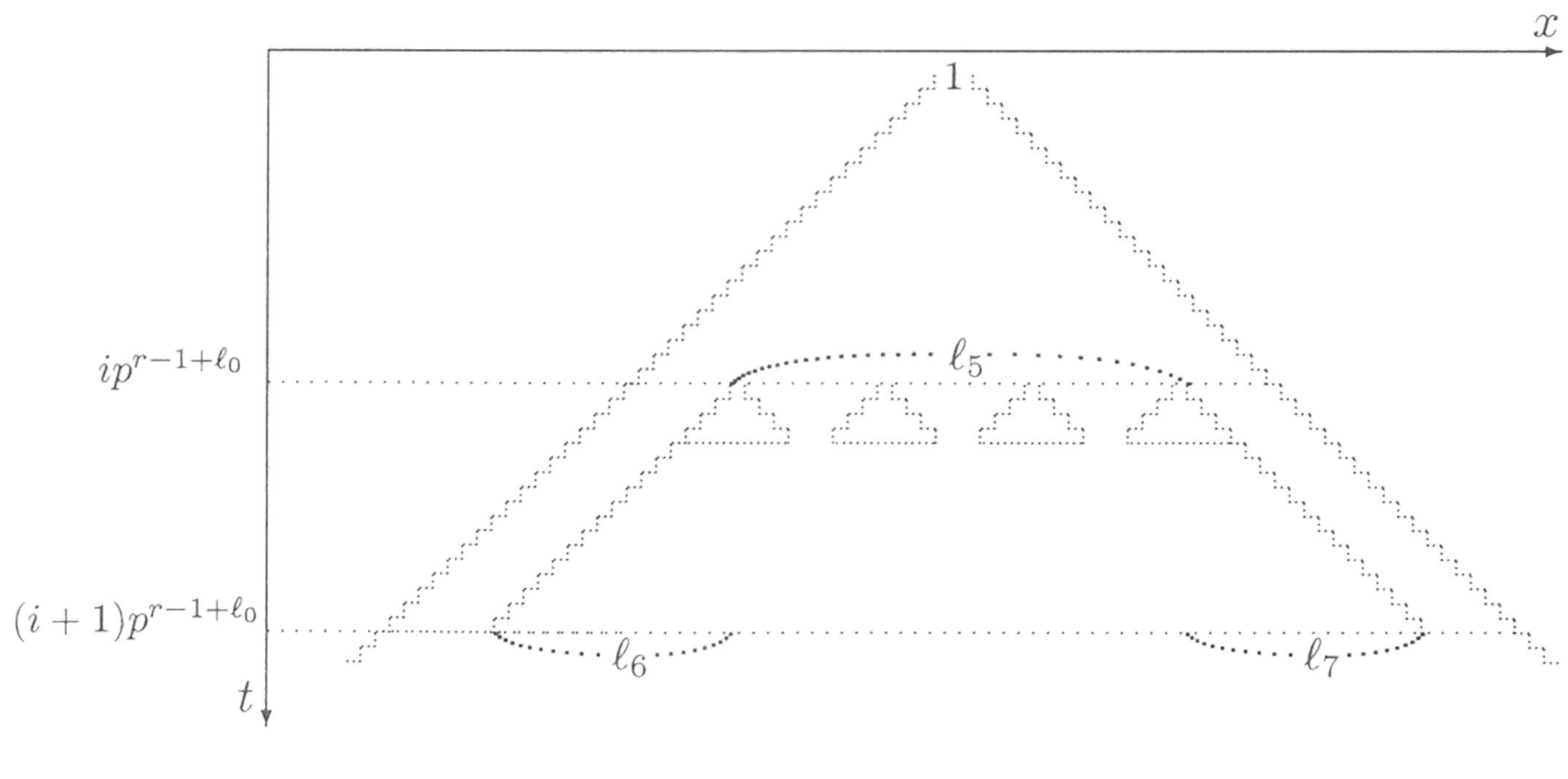
$$\begin{aligned} K_{\boldsymbol{x}} = \{x' \in K(i_0) \mid\ & 0 < t - i_0 p^{r-1+\ell_0} \leq p^{r-1+\ell_0}, \\ & - k_+(t - i_0 p^{r-1+\ell_0}) \leq x - x' \leq -k_-(t - i_0 p^{r-1+\ell_0})\}. \end{aligned}$$

Then we have

$$\begin{aligned} &(\psi_n(\delta))(x/p^n, t/p^n) \\ &= \begin{cases} \sum_{x' \in K_{\boldsymbol{x}}} L^{i_0 p^{r-1+\ell_0}}\delta(x') L^{t - i_0 p^{r-1+\ell_0}}(x - x') & K_{\boldsymbol{x}} \neq \emptyset, \\ 0 & K_{\boldsymbol{x}} = \emptyset \end{cases} \end{aligned} \tag{4.15}$$



(a) $\ell_1 = p^{\ell_0} - 1$, $\ell_2 = |k_- t_0|$ and $\ell_3 = |k_+ t_0|$. The regions A and B is disjoint by the definition of ℓ_0.



(b) $\ell_5 = m_0 p^{\ell_0} + 1$, $\ell_6 = |k_+ p^{r-1+\ell_0}|$ and $\ell_7 = |k_- p^{r-1+\ell_0}|$.

Figure 6: The sketch of space-time pattern of LCA.

for $n > r-1+\ell_0$. Put

$$m_n = \max\{\sharp K_{\boldsymbol{x}} \mid \boldsymbol{x} = (x/p^n, t/p^n) \in X_n \cap X_0\} \tag{4.16}$$

for $n > r-1+\ell_0$. By the definition of $K_{\boldsymbol{x}}$, it is easy to show that there exists $m_0 \in \mathbb{Z}_+$ such that $m_n < m_0$ for all $n > r-1+\ell_0$. Put

$$\begin{aligned} M = \max\{&k_0 p^{r-1+\ell_0} + m_0 p^{\ell_0} + 1, \\ &\sqrt{(|k_+|p^{r-1+\ell_0} + m_0 p^{\ell_0} + 1)^2 + p^{2(r-1+\ell_0)}}, \\ &\sqrt{(|k_-|p^{r-1+\ell_0} + m_0 p^{\ell_0} + 1)^2 + p^{2(r-1+\ell_0)}}\} \text{ (see Figure 6 (b))}, \end{aligned}$$

where k_0, k_+ and k_- are defined as (4.1). For any $\epsilon > 0$, we choose $N > r-1+\ell_0$ satisfying

$$\epsilon > M/p^{N+r-1}.$$

Put $U_\epsilon(\boldsymbol{x}) = \{\boldsymbol{y} \in X_n \cap X_0 \mid d(\boldsymbol{x}, \boldsymbol{y}) < \epsilon\}$. Suppose $n > N$ and $\boldsymbol{x} = (x/p^n, t/p^n) \in X_n \cap X_0$. If $t = 0$, then $H_n(\boldsymbol{x}) = (\psi_n(\delta))(\boldsymbol{x})$. So we consider the case of $t > 0$.

For $i \in \mathbb{Z}_+$ satisfying $ip^{r-1+\ell_0} < t \leq (i+1)p^{r-1+\ell_0}$, suppose $L^{ip^{r-1+\ell_0}}\delta(x') = 0$ for all $(x'/p^n, ip^{r-1+\ell_0}/p^n) \in U_\epsilon(\boldsymbol{x})$. Then $(\psi_n(\delta))(\boldsymbol{x}) = 0$.

Suppose $L^{ip^{r-1+\ell_0}}\delta(x_0)/p \notin \mathbb{N}$ for some $(x_0/p^n, ip^{r-1+\ell_0}/p^n) \in U_\epsilon(\boldsymbol{x})$. Then we have $H_n^{-1}(k) \cap U_\epsilon(\boldsymbol{x}) \neq \emptyset$ for all $k \in \{0, 1, \ldots, p^r - 1\}$ by Proposition 4.3.

Suppose $L^{ip^{r-1+\ell_0}}\delta(x')/p \in \mathbb{N}$ for all $(x'/p^n, ip^{r-1+\ell_0}/p^n) \in U_\epsilon(\boldsymbol{x})$. Put $h_0 = \min\{h \mid L^{ip^{r-1+\ell_0}}\delta(x') = kp^h$ for $k \in \{1, 2, \ldots, p^{r-h}\}$ and $k/p \notin \mathbb{N}\}$. We have $H_n^{-1}(kp^{h_0}) \cap U_\epsilon(\boldsymbol{x}) \neq \emptyset$ for all $k \in \{0, 1, \ldots, p^{r-h_0} - 1\}$ by Proposition 4.3. In the other hand, there exists $k \in \{0, 1, \ldots, p^{r-h_0} - 1\}$ such that $(\psi_n(\delta))(\boldsymbol{x}) = kp^{h_0}$ by (4.15).

So we obtain $D_{0,r}(\overline{H_n^{-1}(j)}, \overline{(\psi_n(\delta))^{-1}(j)}) < \epsilon$ for $n > N$. □

4.3 The definitions of $\{E_\gamma\}$ and $\{A_{b,j,s}\}$

We shall divide X_0 into subsets $\{E_\gamma\}$ and $\{A_{b,j,s}\}$ as follows (see Figure 7 and 8). Let

$$\Gamma = \{(1, j, s) \mid 1 \leq s \leq p^r k_0, 1 \leq j \leq s\} \bigcup \{(2, j, s) \mid 2 \leq s \leq p^r k_0, 1 \leq j \leq s-1\}.$$

We define $\{E^r_\gamma\}(\gamma \in \Gamma)$ as follows:

In case of $\gamma = (1, j, s) \in \Gamma$, let

$$E^r_\gamma = \{(y, q) \mid \frac{s-1}{p^r k_0} \le q \le \frac{s}{p^r k_0},\ -k_+ q + \frac{j-1}{p^r} \le y \le -k_- q - \frac{s-j}{p^r}\};$$

in case of $\gamma = (2, j, s) \in \Gamma$

$$E^r_\gamma = \{(y, q) \mid \frac{s-1}{p^r k_0} \le q \le \frac{s}{p^r k_0},\ -k_- q - \frac{s-j}{p^r} \le y \le -k_+ q + \frac{j}{p^r}\}.$$

Let for $1 \le s \le k_0$ and $1 \le j \le s$,

$$A^r_{1,j,s} = \{(y, q) \mid \frac{s-1}{p^{r-1} k_0} \le q \le \frac{s}{p^{r-1} k_0},\ -k_+ q + \frac{j-1}{p^{r-1}} \le y \le -k_- q - \frac{s-j}{p^{r-1}}\}$$

and for $2 \le s \le k_0$ and $1 \le j \le s-1$,

$$A^r_{2,j,s} = \{(y, q) \mid \frac{s-1}{p^{r-1} k_0} \le q \le \frac{s}{p^{r-1} k_0},\ -k_- q - \frac{s-j}{p^{r-1}} \le y \le -k_+ q + \frac{j}{p^{r-1}}\}.$$

Then we have the following properties.

Proposition 4.8. (1) *The sets $\{E^r_\gamma\}$ have the following properties.*

E-1) *For $\gamma = (b, j, s), \gamma' = (b, j', s) \in \Gamma$, E^r_γ is the shift of $E^r_{\gamma'}$ in the the first coordinate direction for any s and $b \in \{1, 2\}$.*

E-2) $(E^r_\gamma)^\circ \cap (E^r_{\gamma'})^\circ = \emptyset$ *if* $\gamma \ne \gamma'$.

E-3) *If $(S_\ell(X_0))^\circ \cap (S_{\ell'}(X_0))^\circ \ne \emptyset$, then $S_\ell(X_0) \cap S_{\ell'}(X_0)$ is the union of some E^r_γ's.*

E-4) $X_0 = \bigcup_{\gamma \in \Gamma} E^r_\gamma$.

(2) *The sets $\{A^r_{b,j,s}\}_{b,j,s}$ have the following properties.*

A-1) *For any $A^r_{b,j,s}$, there exist $\gamma \in \Gamma$ and $\ell \in \{1, \dots, r_p\}$ such that $A^r_{b,j,s} = S_\ell^{-1}(E^r_\gamma)$.*

A-2) $X_0 = \bigcup_{b=1}^2 \bigcup_{s=1}^{k_0} \bigcup_{j=1}^s A^r_{b,j,s}$.

A-3) $(A^r_{b,j,s})^\circ \cap (A^r_{b',j',s'})^\circ = \emptyset \quad$ *if* $\quad (b, j, s) \ne (b', j', s')$.

Proof. By the definition, we can easily get the result. □

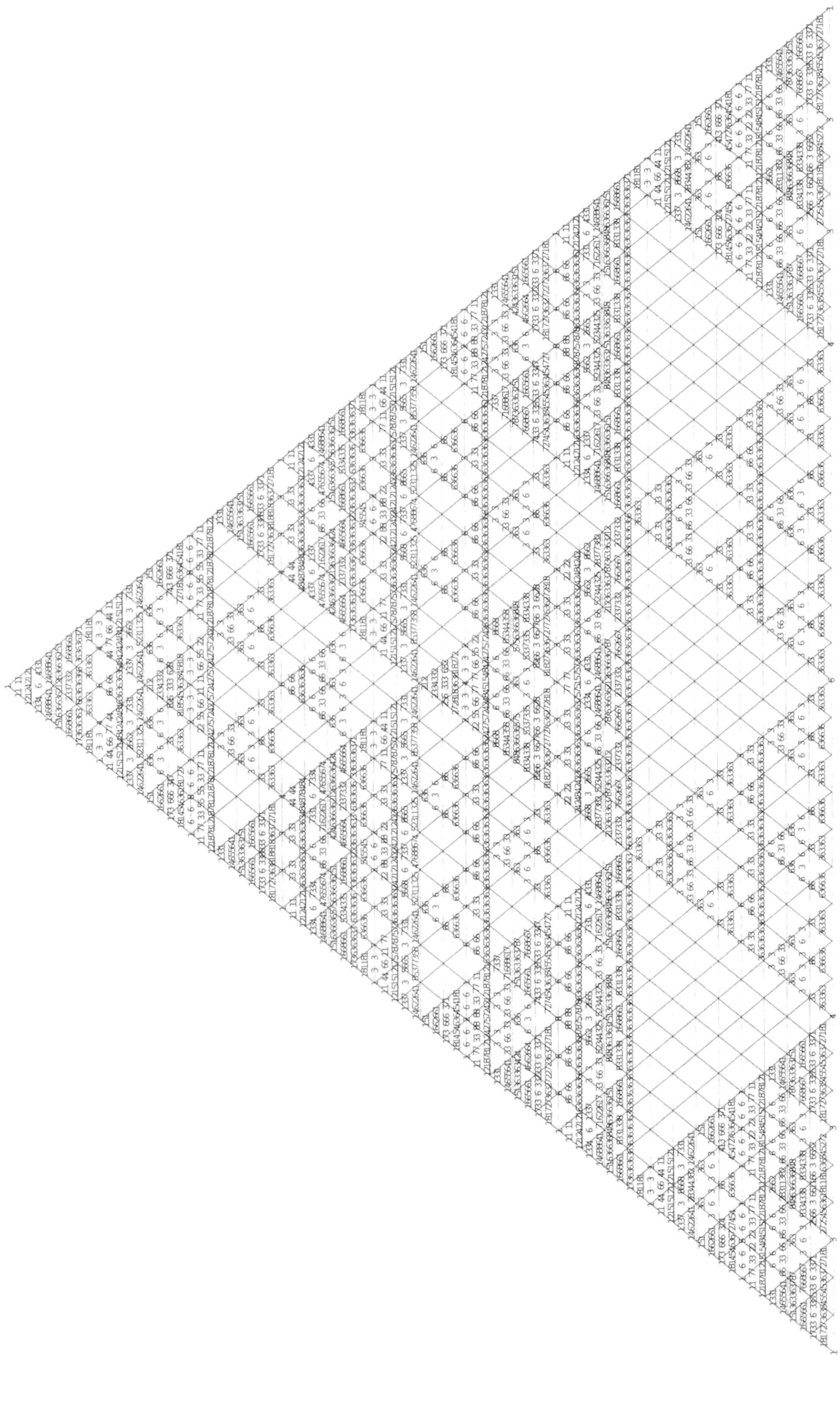


Figure 7: $\{E_\gamma f_\gamma\}_\gamma$ for $(La)(x) = a(x-2) + a(x-1) + a(x+1) + a(x+2) \pmod{3^2}$

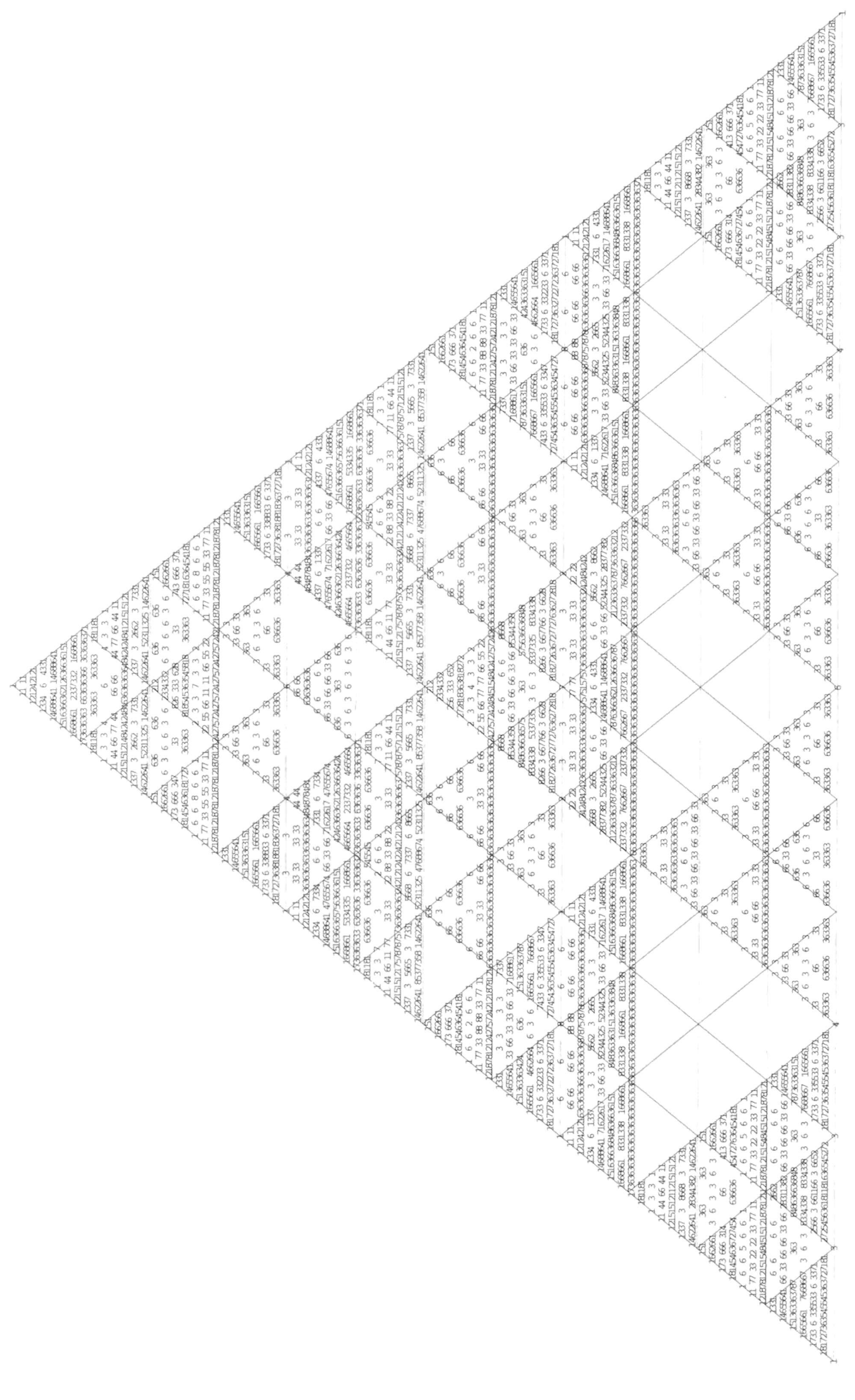


Figure 8: $\{A_{b,j,s}\}_{b,j,s}$ for $(La)(x) = a(x-2) + a(x-1) + a(x+1) + a(x+2) \pmod{3^2}$

4.4 The definition of $\{h_v^n\}$ and their fundamental properties

We shall define the function h_v^n as follows. Put

$$\begin{aligned} V =&\{v = (\gamma_1, \ldots, \gamma_m) \mid m \in \mathbb{Z}_+, \gamma_1 \in \Gamma \text{ with } \#\Lambda_{\gamma_1} \geq 1, \gamma_k \in \Gamma \\ &\text{with } \#\Lambda_{\gamma_k} \geq 1 \text{ and } S_{\ell_{\gamma_{k-1}}}(E^r_{\gamma_k}) \subset E^r_{\gamma_{k-1}} \text{ for any } k \in \{2, \ldots, m\}\}. \end{aligned}$$

For $v = (\gamma_1, \ldots, \gamma_m) \in V$ and $n \in \mathbb{Z}_+$, define

$$\begin{aligned} h_v^n(y, q) = &\sum_{\ell_1 \in \Lambda_{\gamma_1}} \cdots \sum_{\ell_m \in \Lambda_{\gamma_m}} c_{\ell_1} \ldots c_{\ell_m} \\ &\times H_n(S_{\ell_m}^{-1} \ldots S_{\ell_1}^{-1} S_{\ell_{\gamma_1}} \ldots S_{\ell_{\gamma_m}}(y, q)) 1_{S^{-1}_{\ell_{\gamma_m}}(E^r_{\gamma_m})}(y, q) \end{aligned} \tag{4.17}$$

for $(y, q) \in \mathbb{R} \times [0, 1]$.

When $v = (\gamma)$, h_v^n satisfies

$$h_v^n(y, q) = \sum_{\ell \in \Lambda_\gamma} c_\ell H_n(S_\ell^{-1} S_{\ell_\gamma}(y, q)) 1_{S^{-1}_{\ell_\gamma}(E^r_\gamma)}(y, q),$$

and

$$h_v^n(S^{-1}_{\ell_\gamma}(y, q)) = H_n(y, q) 1_{E^r_\gamma}(y, q)$$

for $n \in \mathbb{Z}_+$. Since the length of v is one, h_v^n has the relation with H_{n+1}.

If the length of v is m, then h_v^n has the relation with H_{n+m} and this is useful in estimating the metric $d_f(h_v^n, h_v^{n'})$ as shown in the following lemma.

Lemma 4.9. *For $v = (\gamma_1, \gamma_2, \ldots, \gamma_m) \in V$, $k \in \{1, \ldots, m\}$ and $(y, q) \in \mathbb{R} \times [0, 1]$, put*

$$F_k(y, q) = S_{\ell_{\gamma_1}}(S_{\ell_{\gamma_2}}(\ldots(S_{\ell_{\gamma_k}}(y, q))\ldots)).$$

Then we have

(1) *for $(y, q) \in F_{m-1}(E^r_{\gamma_m})$,*

$$h_v^n(F_m^{-1}(y, q)) = H_{n+m}(y, q) 1_{F_{m-1}(E^r_{\gamma_m})}(y, q)$$

and

(2) *if the sets* $\{j \in \{1,\dots,p-1\} \mid (h_v^n)^{-1}(j) = \emptyset\}$ *and* $\{j \in \{1,\dots,p-1\} \mid (h_v^{n'})^{-1}(j) = \emptyset\}$ *are the same, then*

$$d_f(h_v^n, h_v^{n'}) = p^m d_f(H_{n+m} 1_{F_{m-1}(E^r_{\gamma_m})}, H_{n'+m} 1_{F_{m-1}(E^r_{\gamma_m})})$$

for any $n,\ n' \in \mathbb{Z}_+$.

Proof. The proof is a similar to that of Lemma 4.3 in [3]. □

In a similar way to Proposition 4.4 in [3], we can show the following proposition, which means that the sets $\{j \in \{1,\dots,p-1\} \mid (h_v^n)^{-1}(j) = \emptyset\}$ and $\{j \in \{1,\dots,p-1\} \mid (h_v^{n'})^{-1}(j) = \emptyset\}$ are the same for sufficiently large n, n'.

Proposition 4.10. *For sufficiently large* $n \in \mathbb{Z}_+$, *the following assertions are equivalent for any* $v = (\gamma_1, \dots, \gamma_{m_v}) \in V$, $\ell \in \mathbb{Z}_{p^r} \backslash \{0\}$.

(1) $(H_{n+m_v})^{-1}(\ell) \cap (F_{m_v - 1}(E^r_{\gamma_{m_v}}))^\circ \neq \emptyset$.

(2) $(H_{n+m_v+1})^{-1}(\ell) \cap (F_{m_v - 1}(E^r_{\gamma_{m_v}}))^\circ \neq \emptyset$.

4.5 The definition of $\{M_0^{n,n'}\}$ and their properties

By using h_v^n, we shall define $M_0^{n,n'}$ by

$$M_0^{n,n'} = \sup\{d_f(h_v^n, h_v^{n'}) \mid v \in V\}.$$

Then we have the following

Proposition 4.11. (1) $\sup\{M_0^{n,n'} \mid n, n' \in \mathbb{Z}_+\} < \infty$.

(2) $d_f(H_{n+1}, H_{n'+1}) \leq \frac{1}{p} M_0^{n,n'}$ *holds for sufficiently large* $n, n' \in \mathbb{Z}_+$.

Proof. By using Lemma 4.9 and Proposition 4.10, we get the conclusion in a similar way to [3, Proposition 4.5]. □

Proposition 4.12. *For sufficiently large* n, n', *we have*

$$M_0^{n+1,n'+1} \leq \frac{1}{p} M_0^{n,n'}.$$

Proof. The proof is a similar to that of Proposition 4.6 in [3].

□

4.6 Proof of Theorem 4.1

By using above propositions, we shall prove Theorem 4.1.

(1) By Propositions 4.11 (2) and 4.12, we have

$$\lim_{n,m\to\infty} M_0^{n,m} = 0.$$

By Proposition 4.11 (1) , we have

$$d_f(H_{n+1}, H_{m+1}) \leq \frac{1}{p} M_0^{n,m}.$$

Since we have $d_f(H_n, \psi_n(\delta)) \to 0$ as $n \to \infty$ by Proposition 4.2, we obtain the conclusion.

(2) Since $\{\psi_n(\delta)\} \subset USC|_{X_0}$, we get the result from (1) and Theorem 3.1. $\square$

4.7 Convergence of $\psi_n(a)(a \in \mathcal{P})$ in case of $\mathbb{R} \times [0, 1]$

We consider convergence of $\psi_n(a)(a \in \mathcal{P})$ in a similar way to $\psi_n(\delta)$. We define a function H'_n as follows. For $n \in \{r, r+1, r+2, \ldots\}$ let $X'_n = \{(\frac{x}{p^n}, \frac{jp^{r-1}}{p^n}) \mid x \in \mathbb{Z}, j = 0, 1, \ldots, p^{n-r+1}\}$ and

$$H'_n = \psi_n(a) 1_{X'_n}. \tag{4.18}$$

Then we can show the following theorem in a similar way to the proof of Proposition 4.2.

Proposition 4.13. *For the pseudodistance D_0 on $\mathbb{R} \times [0, 1]$, we have*

$$D_0({H'_n}^{-1}[s+], (\psi_n(a))^{-1}[s+]) \to 0 \text{ as } n \to \infty \text{ for } s \in \{1, \ldots, p^r - 1\}$$

and

$$D_0(\overline{{H'_n}^{-1}(j)}, \overline{(\psi_n(a))^{-1}(j)}) \to 0 \text{ as } n \to \infty \text{ for } j \in \{1, \ldots, p^r - 1\}.$$

So we can show the following theorem in a similar way to Theorem 4.1.

Theorem 4.14. *Let the set G in (2.1) with mod p^r have at least two prime elements. For a nonzero $a \in \mathcal{P}$, we have*

(1) $d_f(\psi_n(a), \psi_m(a)) \to 0$ *as* $n, m \to \infty$.

(2) *Put* $f_a = \bigwedge_{k \geq 1} \bigvee_{n \geq k} \psi_{n+r-1}(a)$, *where* $\bigwedge$ *and* $\bigvee$ *are lattice operations in* USC. *Then we have*

$$D_f(\psi_n(a), f_a) \to 0 \qquad as \qquad n \to \infty.$$

Proof. (1) For H'_n, we can show the following relation in a similar way to the proof of $\psi_n(a)$ in case of mod p.

$$d_f(H'_n, H'_m) \to 0$$

as $n, m \to \infty$. So by Proposition 4.13, we have (1).

(2)We get the result from (1) and Theorem 3.1. □

5 The relation between the limit function and the limit set

In this section, we investigate the relation between the limit function and the limit set of $\{K^f(n, \delta)/p^n\}_n$, which Takahashi defined in [5]. Put

$$K^f(n, \delta) = \{(x, t) \in \mathbb{Z} \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ L^t\delta(x) \not\equiv 0 \pmod{p^f}\}$$

for $f \in \{1, 2, \ldots, r\}$ and

$$K_b(n, \delta) = \{(x, t) \in \mathbb{Z} \times \mathbb{Z}_+ \mid 0 \leq t \leq p^n - 1,\ L^t\delta(x) \equiv b \pmod{p^r}\}$$

for $b \in \{1, 2, \ldots, p^r - 1\}$.

Then the following lemma holds.

Lemma 5.1. [5] *Let* L *be defined as* (2.1) *with mod* p^r *and suppose that at least two elements of* G *is prime and* $f \in \{1, \ldots, r\}$. *Then for* $b \in \mathbb{Z}_{p^r}$ *satisfying* $b/p^{f-1} \in \mathbb{N}$ *and* $b/p^f \notin \mathbb{N}$, *we have*

$$\overline{\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \frac{K_b(n, \delta)}{p^n}} = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n \geq k} \frac{K_b(n, \delta)}{p^n}} = \overline{\bigcup_{k=1}^{\infty} \bigcap_{n \geq k} \frac{K^f(n, \delta)}{p^n}} = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n \geq k} \frac{K^f(n, \delta)}{p^n}}.$$

We first show the relation between $Y_f = \bigcap_{k=1}^{\infty} \overline{\bigcup_{n\geq k} K^f(n,\delta)/p^n}$ (Theorem 5.2) and $\lim_{n\to\infty} \psi_n(\delta)$.

Let $\hat{g}$ be the upper envelope of g, that is,

$$\hat{g}(x,t) = \inf\{\phi(x,t) | \phi \in USC, \phi(x,t) \geq g(x,t)\}.$$

Then the limit function g_a in the pointwise topology (Theorem 2.3) has the relation with a limit set in the sense of Kuratowski limit.

Theorem 5.2. *Suppose the set G in* (2.1) *has at least two prime elements. Let the function g_δ be defined by $g_\delta(y,q) = \lim_{n\to\infty}(\psi_n(\delta))(y,q)$.*

Then

$$\hat{g}_\delta = \sum_{1\leq f\leq r} (p^{r+1-f}-1)p^{f-1} 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i} \tag{5.1}$$

and

$$\hat{g}_\delta = \bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} \psi_{n+r-1}(\delta). \tag{5.2}$$

Proof. For $f \in \{1,2,\ldots,r\}$, let $(y,q) \in Y_f \setminus \cup_{i=1}^{f-1} Y_i$. Then there exists a sequence $\{(y_{n_j}, q_{n_j}) \in K^f(n_j,\delta)/p^{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j\to\infty} n_j = \infty$ and $\lim_{j\to\infty}(y_{n_j}, q_{n_j}) = (y,q)$. Since $g_\delta(y_{n_j}, q_{n_j}) \neq 0$, there exists a sequence $\{(y'_{n_j}, q'_{n_j}) \in K^f(n_j,\delta)/p^{n_j}\}$ such that $g_\delta(y'_{n_j}, q'_{n_j}) = (p^{r+1-f}-1)p^{f-1}$ and $\lim_{j\to\infty}(y'_{n_j}, q'_{n_j}) = (y,q)$. So $\hat{g}_\delta(y,q) = (p^{r+1-f}-1)p^{f-1}$. If $(y,q) \notin Y_f$ for all $f \in \{1,2,\ldots,r\}$, then there exists a neighborhood U of (y,q) and k such that $U \cap K^f(n,\delta)/p^n = \emptyset$ for any $n \geq k$. So $\hat{g}_\delta(y,q) = 0$. Therefore we obtain the equation (5.1).

In order to verify (5.2), we will show

$$\begin{aligned} &\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} (\psi_{n+r-1}(\delta))(y,q) = \\ &\quad \begin{cases} (p^{r+1-f}-1)p^{f-1} & \text{for } (y,q) \in Y_f \setminus \cup_{i=1}^{f-1} Y_i \text{ with } f \in \{1,\ldots,r\}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{5.3}$$

The equation $\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k}(\psi_{n+r-1}(\delta))(y,q) = (p^{r+1-f}-1)p^{f-1}$ holds if and only if

(i) for any $k \in \mathbb{Z}_+$ and $\epsilon > 0$ there exist $(y', q') \in \mathbb{R} \times [0, 1]$ and $n' \geq k$ such that $|(y', q') - (y, q)| < \epsilon$ and $(\psi_{n'+r-1}(\delta))(y', q') > (p^{r+1-f} - 1)p^{f-1} - \epsilon$

and

(ii) for any $\epsilon > 0$ there exists $k \in N$ and a neighborhood U of (y, q) such that $(\psi_{n+r-1}(\delta))(y', q') < (p^{r+1-f} - 1)p^{f-1} + \epsilon$ for all $n \geq k$ and all $(y', q') \in U$.

For $f \in \{1, 2, \ldots, r\}$ and $b \in \mathbb{N}$ satisfying $b/p^{f-1} \in \mathbb{N}$ and $b/p^f \notin \mathbb{N}$, let $(y, q) \in Y_f$. Then for any $\epsilon > 0$ there exists $\{(y_n, q_n) \in K_b(n, \delta)/p^n\}_{n \in \mathbb{Z}_+}$ such that $|(y_n, q_n) - (y, q)| < \epsilon$ by the definition of Y_f and Lemma 5.1. If $(y, q) \notin \cup_{i=1}^{f-1} Y_i$, then for each $i \in \{1, \ldots, f\}$, there does not exist a sequence $\{(y_n, q_n) \in K_b(n, \delta)/p^n\}_{n=1}^{\infty}$ converging to (y, q), where $b/p^{f-1-i} \in \mathbb{N}$ and $b/p^{f-i} \notin \mathbb{N}$. By using the fact above, we obtain (5.3). $\square$

Theorem 5.3. *Suppose that the set G in (2.1) has at least two prime elements. For $a \in \mathcal{P}$ with $a(0) = kp^l$ for $k/p \notin \mathbb{Z}_+$ and $l \in \{0, 1, \ldots, r-1\}$. Put $g_a(y, q) = \lim_{n \to \infty} (\psi_n(a))(y, q)$.*

Then

$$\hat{g}_a = \sum_{1 \leq f \leq r-l} (p^r - p^{f-1+l}) 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i}. \tag{5.4}$$

Proof. For $(y, q) \in Y_f \setminus \cup_{i=1}^{f-1} Y_i$, there exists a sequence $\{(y_{n_j}, q_{n_j}) \in K^f(n_j, \delta)/p^{n_j}\}_{j=1}^{\infty}$ such that $\lim_{j \to \infty} n_j = \infty$, $\lim_{j \to \infty} (y_{n_j}, q_{n_j}) = (y, q)$ and

$$\begin{aligned} g_a(y_{n_j}, q_{n_j}) &= a(0) g_\delta(y_{n_j}, q_{n_j}) \\ &= kbp^{l+f-1} \end{aligned}$$

for $1 \leq b \leq p^{r-f+1} - 1$ and $b/p \notin \mathbb{Z}_+$ by Lemma 5.1 and Theorem 2.3 (3). We have

$$\begin{aligned} &\{bkp^{l+f-1} \pmod{p^r} \mid 1 \leq b \leq p^{r-l-f+1}\} \\ &\qquad = \{bp^{l+f-1} \pmod{p^r} \mid 1 \leq b \leq p^{r-l-f+1}\} \end{aligned}$$

by $k/p \in \mathbb{Z}_+$. So there exists $b \in \{1, \ldots, p^{f+r-1}\}$ such that

$$kbp^{l+f-1} \equiv p^r - p^{f+l-1} \pmod{p^r}.$$

Therefore there exists a sequence $\{(y'_{n_j}, q'_{n_j}) \in K^f(n_j, \delta)/p^{n_j}\}_{j=1}^{\infty}$ such that

$$\lim_{j\to\infty} n_j = \infty,$$
$$\lim_{j\to\infty} (y'_{n_j}, q'_{n_j}) = (y, q)$$

and

$$g_a(y'_{n_j}, q'_{n_j}) = p^r - p^{f-1+l}.$$

There exists a neighborhood U of (y, q) such that $g_a(y', q') \leq p^r - p^{f+l-1}$ for all $(y', q') \in U$ by $(y, q) \notin \cup_{i=1}^{f-1} Y_i$. So $\hat{g}_a(y, q) = p^r - p^{f-1+l}$.

If $(y, q) \notin Y_f$ for all $f \in \{1, 2, \ldots, r\}$, then there exists a neighborhood U of (y, q) and k such that $U \cap K^f(n, a)/p^n = \emptyset$ for any $n \geq k$. So $\hat{g}_a(y, q) = 0$. Therefore we obtain the conclusion. □

For $a = a(x) \in \mathcal{P}$, put

$$G_a = \{x \in \mathbb{Z} \mid a(x) \neq 0\}.$$

Let $\tau_x : \mathcal{P} \to \mathcal{P}$ be a shift operator such that

$$\tau_x a(y) = a(y - x).$$

The following theorem shows the relation between $\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} \psi_{n+r-1}(a)$ and g_a in Theorem 2.3. While the upper envelope of g_a depends on only the value $a(0)$, $\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} \psi_{n+r-1}(a)$ depends on all values $a(x)(x \in \mathbb{Z})$. So g_a is not necessarily equal to $\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} \psi_{n+r-1}(a)$.

Theorem 5.4. *Suppose that the set G in (2.1) has at least two prime elements. Suppose that $a \in \mathcal{P}$ is nonzero and put $g_a(y, q) = \lim_{n\to\infty}(\psi_n(a))(y, q)$. Then*

$$\bigwedge_{k=1}^{\infty} \bigvee_{n\geq k} \psi_{n+r-1}(a) = \bigvee_{x\in G_a} \hat{g}_{\tau_x(a)}.$$

Proof. Let $l_x \in \mathbb{Z}_+$ satisfy $\tau_x(a)(0) = kp^{l_x}(k/p \notin \mathbb{Z}_+)$ and $x_0 \in \mathbb{Z}$ satisfy $l_{x_0} \leq l_x$ for all $x \in \mathbb{Z}$. Since we have

$$\hat{g}_{\tau_{x_0}(a)} = \bigvee_{x\in G_a} \hat{g}_{\tau_x(a)} = \sum_{1\leq f\leq r-l_{x_0}} (p^r - p^{f-1+l_{x_0}}) 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i}$$

by $\hat{g}_{\tau_x(a)} = \sum_{1 \le f \le r - l_x} (p^r - p^{f-1+l_x}) 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i}$, we shall show

$$\bigwedge_{k=1}^{\infty} \bigvee_{n \ge k} \psi_{n+r-1}(a) = \sum_{1 \le f \le r - l_{x_0}} (p^r - p^{f-1+l_{x_0}}) 1_{Y_f \setminus \cup_{i=1}^{f-1} Y_i}.$$

In order to verify it, we show

$$\bigwedge_{k=1}^{\infty} \bigvee_{n \ge k} \psi_{n+r-1}(a) = \begin{cases} p^r - p^{f-1+l_{x_0}} & (y,q) \in Y_f \setminus \cup_{i=1}^{f-1} Y_i, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5}$$

For any $n \in \mathbb{Z}_+$ and $(y,q) \in \mathbb{R} \times [0,1]$, the equation $\psi_{n+r-1}(\tau_x(a))(y,q) = \psi_{n+r-1}(a)(y - x/p^{n+r-1}, q)$ holds. Using the relation above, we can show the equation (5.5) in a similar way to the proof of the equation (5.2) in Theorem 5.2. □

References

[1] C. Kuratowski, Topologie I (2nd ed.), Monografie Matematyczne Vol. 20 (PWN, Warsaw, 1948), 241–250.

[2] M. Matsuto, *A note on the property of linear cellular automata.* Natur. Sci. Rep. Ochanomizu Univ. **50** (1999), 9–15.

[3] M. Matsuto and F. Takeo, *Convergence to the limit set of linear cellular automata.* SUT Journal of Mathematics (to appear).

[4] G. Salinetti and R. J.-B. Wets, *On the convergence of sequences of convex sets in finite dimensions.* SIAM Review 21 (1979), 18 –33.

[5] S. Takahashi, *Self-similarity of linear cellular automata.* J. Comput. System Sci. **44** (1992), 114–140.

[6] S. Willson, *Cellular automata can generate fractals.* Discrete Appl. Math. **8** (1984), 91–99.