Chapter 2

Dynamics of the Dissipative Jaynes-Cummings Model

A new type of the dissipative Jaynes-Cummings model is presented. Interaction between a composite relevant system and a reservoir is introduced to modulate the coupling constant in the relevant system. A quantal master equation is obtained with use of the time convolutionless (TCL) projection operator formalism and is expanded in terms of eigenstates of the composite relevant system. The resulting basic equations become solvable and can be used irrespective of the interaction strength between the relevant subsystems. Moreover, these equations give the correct equilibrium values in each subspace. Several explicit numerical calculations are done.

2.1 Introduction

Studies on quantum dynamics of a spin 1/2 system coupled with bosons have been an extensive subject for a long time. For instance, Jaynes and Cummings [1] introduced a model composed of a spin 1/2 particle interacting with a single-mode boson. The Jaynes-Cummings model is solvable within the so-called rotating wave approximation. Recently, renewed attention is paid to the model in connection with the micromaser [2].

In many physical systems, however, the relevant coupled system is surrounded by environment (reservoir) which exerts certain effects on it. That is, the relevant coupled system is perturbed by the environmental system which is the origin of fluctuation and dissipation mechanism. In the conventional treatment [3], each subsystem of the composite relevant system is assumed to be surrounded by their respective reservoir. That is, our concern lies in the relevant system whose Hamiltonian is of the form

$$H = H_{\rm I} + H_{\rm II} + H_{\rm I\ II}. \tag{2.1.1}$$

In (2.1.1), $H_{\rm I}$ ($H_{\rm II}$) represents the Hamiltonian of subsystem I (II) whereas $H_{\rm I\ II}$ is the interaction Hamiltonian between the two. Then, the density matrix $\rho(t)$ of the relevant system evolves in time according to

$$\frac{\partial}{\partial t}\rho(t) = \frac{1}{i\hbar}\left[H_{\rm I} + H_{\rm II} + H_{\rm I\ II}\ , \rho(t)\ \right] + (\Gamma_{\rm I} + \Gamma_{\rm II})\,\rho(t)\,. \tag{2.1.2}$$

The first term on the right hand side of (2.1.2) describes coherent motion of the coupled system including the interaction between the subsystems I and II . On the other hand, the relaxation operator $\Gamma_{\rm I}$ ($\Gamma_{\rm II}$) is obtained for the subsystem I (II) alone, that is, when the interaction Hamiltonian $H_{\rm I\ II} = 0$. In this framework, the subsystem I (II) relaxes to the equilibrium specified by the Hamiltonian $H_{\rm I}$ ($H_{\rm II}$). Each subsystem relaxes to its respective equilibrium independently but not to the one for the whole relevant coupled system.

Since we consider strongly coupled systems, effect of the interaction between the subsystems on the relaxation operator can never be neglected. Thus, for the composite relevant system of the Hamiltonian (2.1.1), the the relaxation operator Γ should not

be characterized by H_{I} (H_{II}) but by (2.1.1) itself [4]. Generally speaking, it is quite difficult to derive Γ and solve the resulting equation. And thus, there have been only a small amount of work on this problem. As a special case, detailed study was done on the parametric amplifier [5]. In this chapter we introduce another solvable relaxation model other than the parametric amplifier, i.e., a dissipative Jaynes-Cummings model, relaxation operator of which is determined by (2.1.1). This model enables us to take into account a new type of dissipation mechanism which is out of a scope of the conventional treatment.

2.2 Quantal master equation

Our system is composed of a spin 1/2 particle strongly coupled with a single-mode boson. The Hamiltonian of the system is given by

$$H = H_S + H_b + H_i \tag{2.2.1}$$

where

$$H_S = \hbar\omega_0 S_z, \tag{2.2.2}$$

$$H_b = \hbar\omega_b b^\dagger b \tag{2.2.3}$$

and

$$H_i = \hbar g_{\|} b^\dagger b S_z + \hbar g_\perp \left(b S_+ + b^\dagger S_- \right). \tag{2.2.4}$$

In these expressions, we introduced the spin 1/2 operator $\boldsymbol{S} = (S_x,\ S_y,\ S_z)$, $S_\pm = S_x \pm i S_y$, the Larmor (angular) frequency ω_0, and the annihilation (creation) operator b ($b^\dagger$), the (angular) frequency ω_b for the boson. The first term on the right hand side of (2.2.4) represents the adiabatic interaction which causes dephasing effect whereas the second of (2.2.4) is the non-adiabatic term which gives rise to spin flip with energy exchange. The system represented by (2.2.1)-(2.2.4) is a slightly extended version of the Jaynes-Cummings model in the rotating wave approximation.

In our model, the non-adiabatic coupling constant $g_\perp$ in (2.2.4) is considered to be modified due to an interaction with a reservoir:

$$g_\perp \to g_\perp + g_B \sum_l \left(B_l^\dagger + B_l \right) \tag{2.2.5}$$

where $B_l^\dagger$ and B_l represent the reservoir variables. Here we assume that the reservoir is composed of harmonic oscillators and the Hamiltonian of the reservoir is given by

$$H_B = \hbar \sum_l \omega_l B_l^\dagger B_l \tag{2.2.6}$$

with $\omega_l > 0$. Thus our system (characterized by $H = H_S + H_b + H_i$) is perturbed by the reservoir (characterized by H_B) through the Hamiltonian of the form

$$H_{iB} = \hbar \left(b S_+ + b^\dagger S_- \right) g_B \sum_l \left(B_l^\dagger + B_l \right). \tag{2.2.7}$$

The reduced density matrix of the relevant system is given by

$$\rho\left(t\right) = \mathrm{tr}_B W\left(t\right) \tag{2.2.8}$$

where $W\left(t\right)$ is the density matrix of the whole system including the reservoir. The symbol tr_B stands for the trace operation over the reservoir variables. With use of the time convolutionless (TCL) formalism [6] (See Appendix A), the quantal master equation of the relevant system is given by

$$\frac{\partial}{\partial t}\rho\left(t\right) = \frac{1}{i\,\hbar}\left[H\ ,\rho\left(t\right)\ \right] + \mathit{\Gamma}\ \rho\left(t\right) \tag{2.2.9}$$

where

$$\begin{aligned}
\mathit{\Gamma}\ \rho\left(t\right) &= \sum_l g_B^2 \int_0^t dt' \Big\{ \left(\langle B_l^\dagger\left(t'\right) B_l \rangle_B + \langle B_l\left(t'\right) B_l^\dagger \rangle_B \right) \\
&\times \left(\left[\underline{bS_+}\left(-t'\right)\rho\left(t\right),\ \underline{b^\dagger S_-}\ \right] + \left[\underline{b^\dagger S_-}\left(-t'\right)\rho\left(t\right),\ \underline{bS_+}\ \right] \right) \\
&+ \left(\langle B_l^\dagger\left(-t'\right) B_l \rangle_B + \langle B_l\left(-t'\right) B_l^\dagger \rangle_B \right) \\
&\times \left(\left[\underline{bS_+},\ \rho\left(t\right)\ \underline{b^\dagger S_-}\left(-t'\right) \right] + \left[\underline{b^\dagger S_-},\ \rho\left(t\right)\ \underline{bS_+}\left(-t'\right) \right] \right) \Big\}.
\end{aligned} \tag{2.2.10}$$

In (2.2.10), the operators $B_l(t)$, $B_l^\dagger(t)$, $\underline{bS_+}(t)$ and $\underline{bS_+}(t)$ are respectively given by

$$B_l\left(t\right) = \mathrm{e}^{iH_B t/\hbar}\ B_l\ \mathrm{e}^{-iH_B t/\hbar}, \tag{2.2.11}$$

$$B_l^\dagger\left(t\right) = \mathrm{e}^{iH_B t/\hbar}\ B_l^\dagger\ \mathrm{e}^{-iH_B t/\hbar}, \tag{2.2.12}$$

$$\underline{bS_+}\left(t\right) = \mathrm{e}^{iHt/\hbar}\ \underline{bS_+}\ \mathrm{e}^{-iHt/\hbar}, \tag{2.2.13}$$

and

$$\underline{b^\dagger S_-}\left(t\right) = \mathrm{e}^{iHt/\hbar}\ \underline{b^\dagger S_-}\ \mathrm{e}^{-iHt/\hbar}. \tag{2.2.14}$$

It should be noted that time evolution of the system operators (2.2.13) and (2.2.14) are determined by the total Hamiltonian H of the relevant system, (2.2.1).

2.3 Eigenstates and eigenvalues of the system

Following the method of Jaynes and Cummings [1, 7], eigenstates of the Hamiltonian (2.2.1) are obtained as

$$|\varphi(n,1)\rangle \equiv \cos\theta_n |n+1,-1\rangle + \sin\theta_n |n,+1\rangle , \tag{2.3.1}$$

$$|\varphi(n,2)\rangle \equiv -\sin\theta_n |n+1,-1\rangle + \cos\theta_n |n,+1\rangle , \tag{2.3.2}$$

for $n = 0, 1, ...$ and a state

$$|0,-1\rangle \tag{2.3.3}$$

which is not written in the form of (2.3.1) and (2.3.2). In these expressions, $|n,+1\rangle$ represents the state with n bosons and spin-up whereas $|n,-1\rangle$ spin-down. The eigenstates are orthogonal to one another and satisfy completeness relation of the form

$$\sum_{n=0} \{|\varphi(n,1)\rangle \langle\varphi(n,1)| + |\varphi(n,2)\rangle \langle\varphi(n,2)|\} + |0,-1\rangle \langle 0,-1| = 1. \tag{2.3.4}$$

We have

$$H \ |\varphi(n,1)\rangle = E_+^n \ |\varphi(n,1)\rangle , \tag{2.3.5}$$

$$H \ |\varphi(n,2)\rangle = E_-^n \ |\varphi(n,2)\rangle , \tag{2.3.6}$$

and

$$H \ |0,-1\rangle = E_0 \ |0,-1\rangle . \tag{2.3.7}$$

In these expressions we have put

$$E_\pm^n = \hbar \left\{ \omega_b \left(n + \frac{1}{2} \right) - \frac{g_{||}}{4} \pm \lambda(n) \right\} \tag{2.3.8}$$

and

$$E_0 = -\frac{\hbar\omega_0}{2} \tag{2.3.9}$$

with

$$\lambda(n) = \sqrt{\frac{1}{4}\left(\Delta\omega - \frac{g_{||}(2n+1)}{2} \right)^2 + g_\perp^2 (n+1)}, \tag{2.3.10}$$

$\Delta\omega$ being defined by $\Delta\omega = \omega_b - \omega_0$.

The angle θ_n is determined to be

$$\tan\theta_n = \frac{g_\perp\sqrt{n+1}}{\frac{1}{2}\left(\Delta\omega - \frac{g_\parallel(2n+1)}{2}\right) + \lambda(n)} \tag{2.3.11}$$

for $g_\perp \neq 0$.

For $g_\perp = 0$, we have to put $\tan\theta_n = 0$ irrespective of $\Delta\omega > g_\parallel(2n+1)/2$ or $\Delta\omega < g_\parallel(2n+1)/2$.

2.4 Derivation of basic equations

We denote components of the density matrix $\rho(t)$ in terms of the eigenstates (2.3.1)-(2.3.3) as

$$\rho_{\alpha\beta}^{nm}(t) = \langle \varphi(n,\alpha)| \rho(t) |\varphi(m,\beta)\rangle, \tag{2.4.1}$$

$$\rho_{0\beta}^{m}(t) = \langle 0,-1| \rho(t) |\varphi(m,\beta)\rangle, \tag{2.4.2}$$

$$\rho_{\alpha 0}^{n}(t) = \langle \varphi(n,\alpha)| \rho(t) |0,-1\rangle, \tag{2.4.3}$$

$$\rho_{00}(t) = \langle 0,-1| \rho(t) |0,-1\rangle \tag{2.4.4}$$

with $\alpha,\ \beta = 1,2$. The diagonal components satisfy the normalization condition:

$$\sum_n \{\rho_{11}^{nn}(t) + \rho_{22}^{nn}(t)\} + \rho_{00}(t) = 1. \tag{2.4.5}$$

We expand the quantal master equation (2.2.9) in terms of (2.3.1)-(2.3.3). The corresponding equations of the components of the density matrix are given as

$$\frac{\partial}{\partial t}\rho_{\alpha\beta}^{nm}(t) = \frac{1}{i\hbar}\langle \varphi(n,\alpha)| [H,\rho(t)] |\varphi(m,\beta)\rangle + \langle \varphi(n,\alpha)| \Gamma\rho(t) |\varphi(m,\beta)\rangle, \tag{2.4.6}$$

$$\frac{\partial}{\partial t}\rho_{0\beta}^{m}(t) = \frac{1}{i\hbar}\langle 0,-1| [H,\rho(t)] |\varphi(m,\beta)\rangle + \langle 0,-1| \Gamma\rho(t) |\varphi(m,\beta)\rangle, \tag{2.4.7}$$

$$\frac{\partial}{\partial t}\rho_{\alpha 0}^{n}(t) = \frac{1}{i\hbar}\langle \varphi(n,\alpha)| [H,\rho(t)] |0,-1\rangle + \langle \varphi(n,\alpha)| \Gamma\rho(t) |0,-1\rangle, \tag{2.4.8}$$

$$\frac{\partial}{\partial t}\rho_{00}(t) = \frac{1}{i\hbar}\langle 0,-1| \Gamma\rho(t) |0,-1\rangle. \tag{2.4.9}$$

Next we show manipulation of the second term on the right hand side of (2.4.6). This may serve for illustration:

$$\begin{aligned}
&\langle \varphi(n,\alpha)| \Gamma\rho(t) |\varphi(m,\beta)\rangle \\
&\quad = \sum_l g_B^2 \int_0^t dt' \Big\{ \left(\langle B_l^\dagger(t') B_l\rangle_B + \langle B_l(t') B_l^\dagger\rangle_B \right) \\
&\quad \times \langle \varphi(n,\alpha)| \left[\underline{bS_+}(-t')\rho(t),\ \underline{b^\dagger S_-} \right] + \left[\underline{b^\dagger S_-}(-t')\rho(t),\ \underline{bS_+} \right] |\varphi(m,\beta)\rangle \\
&\quad + \left(\langle B_l^\dagger(-t') B_l\rangle_B + \langle B_l(-t') B_l^\dagger\rangle_B \right) \\
&\quad \times \langle \varphi(n,\alpha)| \left[\underline{bS_+},\ \rho(t)\ \underline{b^\dagger S_-}(-t') \right] + \left[\underline{b^\dagger S_-},\ \rho(t)\ \underline{bS_+}(-t') \right] |\varphi(m,\beta)\rangle \Big\}.
\end{aligned} \tag{2.4.10}$$

In (2.4.10), the relevant operators are written in terms of the eigenbras and kets:

$$\underline{bS_+}\,(t) = \sum_{n=0}\sum_{\alpha,\beta} |\varphi\,(n,\alpha)\rangle\, P^{nn}_{\alpha\beta}\,(t)\,\langle\varphi\,(n,\beta)| \tag{2.4.11}$$

and

$$\underline{b^\dagger S_-}\,(t) = \sum_{n=0}\sum_{\alpha,\beta} |\varphi\,(n,\alpha)\rangle\, Q^{nn}_{\alpha\beta}\,(t)\,\langle\varphi\,(n,\beta)| \tag{2.4.12}$$

where

$$P^{nn}_{11}\,(t) = -P^{nn}_{22}\,(t) = Q^{nn}_{11}\,(t) = -Q^{nn}_{22}\,(t) = \sqrt{n+1}\sin\theta_n\cos\theta_n, \tag{2.4.13}$$

$$P^{nn}_{12}\,(t) = \{Q^{nn}_{21}\,(t)\}^\dagger = -\mathrm{e}^{2i\lambda(n)t}\sin^2\theta_n, \tag{2.4.14}$$

$$P^{nn}_{21}\,(t) = \{Q^{nn}_{12}\,(t)\}^\dagger = -\mathrm{e}^{-2i\lambda(n)t}\cos^2\theta_n \tag{2.4.15}$$

with (2.3.10).

We further assume that the correlation time of the reservoir is much shorter than the relaxation time of the system. In this narrowing limit, we have

$$\begin{aligned}\phi^-_n &\equiv \sum_l g_B^2 \int_0^\infty dt'\,\langle B^\dagger_l\,(t')\,B_l\rangle_B \mathrm{e}^{-2i\lambda(n)t'} \\ &= \sum_l g_B^2 \int_0^\infty dt' \langle B^\dagger_l\,(-t')\,B_l\rangle_B \mathrm{e}^{2i\lambda(n)t'} = \kappa\tilde{n}\,(n)\,,\end{aligned} \tag{2.4.16}$$

$$\begin{aligned}\psi^-_n &\equiv \sum_l g_B^2 \int_0^\infty dt' \langle B_l\,(t')\,B^\dagger_l\rangle_B \mathrm{e}^{-2i\lambda(n)t'} \\ &= \sum_l g_B^2 \int_0^\infty dt' \langle B_l\,(-t')\,B^\dagger_l\rangle_B \mathrm{e}^{2i\lambda(n)t'} = \kappa\,(\tilde{n}\,(n)+1)\end{aligned} \tag{2.4.17}$$

where κ is the damping constant and

$$\tilde{n}\,(n) = \left(\mathrm{e}^{2\hbar\lambda(n)/k_BT} - 1\right)^{-1}, \tag{2.4.18}$$

T being the temperature of the reservoir. All the other correlation functions of B_l and $B^\dagger_l$ vanish identically. For later convenience, we define the following vectors:

$$\boldsymbol{\rho}^{nm}\,(t) \equiv \begin{pmatrix} \rho^{nm}_{11}\,(t) \\ \rho^{nm}_{12}\,(t) \\ \rho^{nm}_{21}\,(t) \\ \rho^{nm}_{22}\,(t) \end{pmatrix}, \tag{2.4.19}$$

$$\boldsymbol{\rho}_0{}^m\,(t) \equiv \begin{pmatrix} \rho_{01}{}^m\,(t) \\ \rho_{02}{}^m\,(t) \end{pmatrix} \tag{2.4.20}$$

and

$$\boldsymbol{\rho}^{n}{}_{0}(t) \equiv \begin{pmatrix} \rho^{n}_{10}(t) \\ \rho^{n}_{20}(t) \end{pmatrix} \tag{2.4.21}$$

for $n, m \geq 0$. After manipulating similarly, we can calculate (2.4.6)-(2.4.9) to give a set of vector equations of the form:

$$\frac{\partial}{\partial t}\boldsymbol{\rho}^{nm}(t) = \{-i\,\mathbf{L}^{nm} + \kappa\,\Gamma(n,m)\}\,\boldsymbol{\rho}^{nm}(t), \tag{2.4.22}$$

$$\frac{\partial}{\partial t}\boldsymbol{\rho}_{0}{}^{m}(t) = \{-i\,\mathbf{L}_{0}{}^{m} + \kappa\,\Gamma(m)\}\,\boldsymbol{\rho}_{0}{}^{m}(t), \tag{2.4.23}$$

$$\frac{\partial}{\partial t}\boldsymbol{\rho}^{n}{}_{0}(t) = \{-i\,\mathbf{L}^{n}{}_{0} + \kappa\,\Gamma(n)\}\,\boldsymbol{\rho}^{n}{}_{0}(t), \tag{2.4.24}$$

$$\frac{\partial}{\partial t}\rho_{00}(t) = 0 \tag{2.4.25}$$

with the coefficient matrices $\mathbf{L}$ and Γ defined in Appendix B.

It is a very characteristic of our basic equations (2.4.22)-(2.4.25) that they are separable with respect to the index n. On the other hand, with the other type of system-reservoir interaction represented by

$$H_{bB} = \hbar\left(b + b^{\dagger}\right) g_{bB} \sum_{l}\left(B_{l}^{\dagger} + B_{l}\right) \tag{2.4.26}$$

we obtain time evolution equations having tridiagonal recurrence terms with respect to index (n, m) and their analytic solution in the form of continued fraction [8]. Indeed we have treated relaxation effect due to the same reservoir, but the effect itself depends on the details of the interaction mechanism represented by (2.2.7) and (2.4.26). That is, the difference depends on a way of operation of the system operators coupling with the reservoir on the eigenstates of the relevant system.

2.5 Quasi-probability density for the boson system

The normally ordered quasi-probability density for the boson system is given by

$$F(z,t) = \langle z| \operatorname{tr}_S \rho(t) |z\rangle \tag{2.5.1}$$

where $|z\rangle$ is the boson coherent state, namely,

$$b|z\rangle = z|z\rangle \tag{2.5.2}$$

and the symbol tr_S stands for the trace operation over the spin variable. The quasi-probability density can be rewritten with use of the eigenstates of H:

$$\begin{aligned} F(z,t) &= \operatorname{tr}_S \left[\sum_{n,m} \sum_{\alpha,\beta} \langle z|\, \varphi(n,\alpha)\rangle\, \rho_{\alpha\beta}^{nm}(t)\, \langle \varphi(m,\beta)|\, z\rangle \right. \\ &+ \sum_{m} \sum_{\beta} \langle z|\, 0,-1\rangle\, \rho_{0\beta}^{m}(t)\, \langle \varphi(m,\beta)|\, z\rangle \\ &+ \sum_{n} \sum_{\alpha} \langle z|\, \varphi(n,\alpha)\rangle\, \rho_{\alpha 0}^{n}(t)\, \langle 0,-1|\, z\rangle \\ &+ \left. \langle z|\, 0,-1\rangle\, \rho_{00}\, \langle 0,-1|\, z\rangle \right]. \end{aligned} \tag{2.5.3}$$

Owing to the relation

$$\langle n|\, z\rangle = \exp\left[-|z|^2/2\right] \frac{z^n}{\sqrt{n!}}, \tag{2.5.4}$$

where $|n\rangle$ is the number state, we have

$$\langle \varphi(n,1)|\, z\rangle = \mathrm{e}^{-|z|^2/2} \frac{z^n}{\sqrt{n!}} \left(\frac{z}{\sqrt{n+1}} \cos\theta_n \langle -1| + \sin\theta_n \langle +1| \right), \tag{2.5.5}$$

and

$$\langle \varphi(n,2)|\, z\rangle = \mathrm{e}^{-|z|^2/2} \frac{z^n}{\sqrt{n!}} \left(-\frac{z}{\sqrt{n+1}} \sin\theta_n \langle -1| + \cos\theta_n \langle +1| \right). \tag{2.5.6}$$

Inserting (2.5.5) and (2.5.6) into (2.5.3) and performing the trace operation, we find

$$F(z,t) = \mathrm{e}^{-|z|^2} \left[\sum_{n,m} \frac{z^{*n} z^m}{\sqrt{n!m!}} \left\{ \left(\sin\theta_n \sin\theta_m + \frac{|z|^2}{\sqrt{(n+1)(m+1)}} \cos\theta_n \cos\theta_m \right) \rho_{11}^{nm}(t) \right. \right.$$

$$
\begin{aligned}
&+ \quad \left(\sin\theta_n \cos\theta_m - \frac{|z|^2}{\sqrt{(n+1)(m+1)}} \cos\theta_n \sin\theta_m \right) \rho_{12}^{nm}(t) \\
&+ \quad \left(\cos\theta_n \sin\theta_m - \frac{|z|^2}{\sqrt{(n+1)(m+1)}} \sin\theta_n \cos\theta_m \right) \rho_{21}^{nm}(t) \\
&+ \quad \left(\cos\theta_n \cos\theta_m + \frac{|z|^2}{\sqrt{(n+1)(m+1)}} \sin\theta_n \sin\theta_m \right) \rho_{22}^{nm}(t) \Bigg\} \\
&+ \quad \sum_n \frac{z^{*n+1}}{\sqrt{(n+1)!}} \left(\cos\theta_n \rho_{10}^n(t) - \sin\theta_n \rho_{20}^n(t) \right) \\
&+ \quad \sum_m \frac{z^{m+1}}{\sqrt{(m+1)!}} \left(\cos\theta_m \rho_{01}^m(t) - \sin\theta_m \rho_{02}^m(t) \right) + \rho_{00}(t) \Bigg] . \qquad (2.5.7)
\end{aligned}
$$

Thus we can extract information on the boson system from the density matrix of the coupled system.

2.6 Calculations of the relaxation process

2.6.1 Initial condition and Expectation values

In order to solve the basic equations derived in the previous sections, we impose the following initial condition. At $t = 0$, the relevant system is assumed to be in a state represented by

$$\rho(0) = |+1\rangle\langle+1| \cdot |n\rangle\langle n|, \tag{2.6.1}$$

namely, the system is in the up-spin and n boson state. The initial density matrix $\rho(0)$ is rewritten in terms of the eigenstates as

$$\begin{aligned} \rho(0) &= \sin^2\theta_n |\varphi(n,1)\rangle\langle\varphi(n,1)| + \cos^2\theta_n |\varphi(n,2)\rangle\langle\varphi(n,2)| \\ &+ \sin\theta_n \cos\theta_n \left(|\varphi(n,1)\rangle\langle\varphi(n,2)| + |\varphi(n,2)\rangle\langle\varphi(n,1)|\right). \end{aligned} \tag{2.6.2}$$

We can calculate expectation value of an arbitrary operator:

$$\langle A\rangle_t = \frac{\mathrm{Tr}\left[\rho(t)\, A\right]}{\mathrm{Tr}\left[\rho(t)\right]}. \tag{2.6.3}$$

For instance, we have

$$\begin{aligned} \langle S_z\rangle_t &= \frac{1}{2}\sum_n \{-\cos 2\theta_n \left(\rho_{11}^{nn}(t) - \rho_{22}^{nn}(t)\right) \\ &+ \sin 2\theta_n \left(\rho_{12}^{nn}(t) + \rho_{21}^{nn}(t)\right) - \rho_{00}(t)\ \}, \end{aligned} \tag{2.6.4}$$

$$\begin{aligned} \langle b^\dagger b\rangle_t &= \sum_n \left\{ \left(n + \cos^2\theta_n\right) \rho_{11}^{nn}(t) \right. \\ &- \cos\theta_n \sin\theta_n \left(\rho_{12}^{nn}(t) + \rho_{21}^{nn}(t)\right) \\ &+ \left. \left(n + \sin^2\theta_n\right) \rho_{22}^{nn}(t) \right\}. \end{aligned} \tag{2.6.5}$$

In order to find such quantities like $\langle S_z\rangle_t$, $\langle b^\dagger b\rangle_t$, $\langle bS_+\rangle_t$, $\langle b^\dagger S_-\rangle_t$, $\langle b^\dagger b S_z\rangle_t$ from the initial condition (2.6.2), it is sufficient to determine time evolution of the diagonal components $\rho_{\alpha\beta}^{nn}$ and ρ_{00}. That is, we have the equations:

$$\frac{\partial}{\partial t}\boldsymbol{\rho}^{nn}(t) = \{-i\ \mathbf{L}^{nn} + \kappa\ \Gamma(n,n)\}\, \boldsymbol{\rho}^{nn}(t), \tag{2.6.6}$$

with (2.4.25).

2.6.2 Numerical calculation

It is interesting to examine relaxation process starting from the vacuum state of the bosons. That is, $\rho(0)$ is given by (2.6.2) with $n = 0$. We made numerical calculations with this initial condition.

In Fig.2.1-a and Fig.2.1-b, we show time evolution of expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ as a function of $\tau = \kappa t$. Time evolution of these two quantities are completely correlated. This is the reflection of energy exchange between the relevant subsystems. The damping rate is dependent on the temperature, but after a long time, all these quantities approach to the correct equilibrium values in the subspace specified by n:

$$\langle S_z \rangle_{eq.n} = -\frac{1}{2} \frac{\cos 2\theta_n \left(\mathrm{e}^{-E_+^n/k_BT} - \mathrm{e}^{-E_-^n/k_BT} \right)}{\mathrm{e}^{-E_+^n/k_BT} + \mathrm{e}^{-E_-^n/k_BT}}, \tag{2.6.7}$$

$$\langle b^\dagger b \rangle_{eq.n} = \frac{\left(n + \cos^2\theta_n \right) \mathrm{e}^{-E_+^n/k_BT} + \left(n + \sin^2\theta_n \right) \mathrm{e}^{-E_-^n/k_BT}}{\mathrm{e}^{-E_+^n/k_BT} + \mathrm{e}^{-E_-^n/k_BT}}. \tag{2.6.8}$$

Entire information on the time evolution of the boson subsystem is in the quasi-probability density $F(z,t)$, (2.5.7). In Figs.2.1-c, we show time evolution of the quasi-probability density for $\tilde{T} = k_BT/\hbar\kappa = 1.0$. At $\tau = 0$, the quasi-probability has the Gaussian distribution centered at $z = 0$. The ring shape of a quasi-probability density of Fig.2.1-c-(2) corresponds to the first peak of the dashed line in Fig.2.1-b and the peaked quasi-probability density of Fig.2.1-c-(3) corresponds to the first valley of Fig.2.1-b. Center of the quasi-probability density oscillates in time synchronizing with the oscillation of $\langle b^\dagger b \rangle_\tau$ and at last, the quasi-probability density tends to be the equilibrium function (Fig.2. 1-c-(4)).

Effect of $\tilde{g}_{||} = g_{||}/\kappa$ is seen in Fig.2.2. It gives shift of the equilibrium values in low temperature. In Fig.2.3 we show time evolution of the system for off-resonant condition $\tilde{\omega}_0 = \omega_0/\kappa > \tilde{\omega}_b = \omega_b/\kappa$. At the low temperature, the equilibrium values are quite sensitively dependent on the values of the parameters. But at the high temperature, the equilibrium value of $\langle S_z \rangle_\tau$ tends to zero, reflecting equi-partition of the two states (2.3.1) and (2.3.2).

In order to investigate characteristic properties of each constituent term of (2.2.7), let us divide the interaction Hamiltonian between the relevant system and the reservoir

into two parts:

$$H_{iB1} = \hbar g_B \sum_l \left(b S_+ B_l^\dagger + b^\dagger S_- B_l \right) \tag{2.6.9}$$

and

$$H_{iB2} = \hbar g_B \sum \left(b S_+ B_l + b^\dagger S_- B_l^\dagger \right). \tag{2.6.10}$$

The physical meaning of the division is that (2.6.9) is compatible with the energy conservation for the case $\omega_b > \omega_0$ and (2.6.10) for the case $\omega_b < \omega_0$ as illustrated in Fig.2.4. The Hamiltonians (2.6.9) and (2.6.10) give the relaxation operators Γ_1 and Γ_2, respectively, where we put $\Gamma = \Gamma_1 + \Gamma_2$ with Γ given by (2.2.10).

We show time evolution of $\langle S_z \rangle_\tau$ by the relaxation operator Γ_1 and Γ_2 in Figs.2.5. When the spin and the boson are in resonance (Fig.2.5-a), time evolution by the two relaxation operators coincides ($\Gamma_1 = \Gamma_2$). When we treat the system under near resonance condition, the relaxation operator (2.6.10) contributes equally well as (2.6.9) and hence we should treat (2.2.7) as a whole. On the other hand, with non-resonant condition (Fig.2.5-b), the damping by the relaxation operator Γ_1 is much slower than that of Γ_2. In this case, the damping is dominated by the process represented by Γ_2 which is compatible with the energy conservation. This is clearly seen in Fig.2.3 where the dumping due to Γ_1 is ineffective.

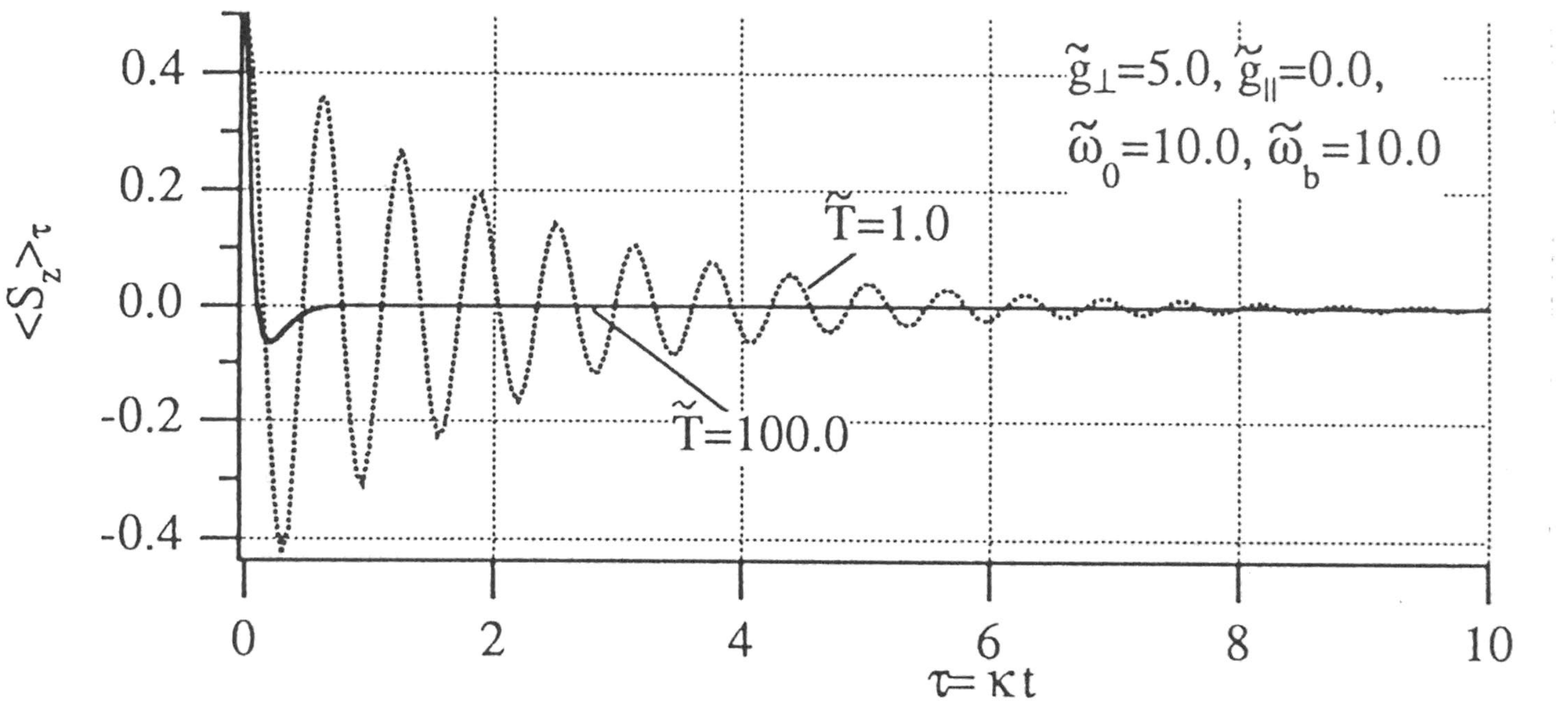


Fig.2.1-a Time evolution of $\langle S_z \rangle_\tau$ as a function of scaled time $\tau = \kappa t$. System parameters are $\tilde{g}_\perp = g_\perp/\kappa = 5.0$, $\tilde{g}_{||} = g_{||}/\kappa = 0$, $\tilde{\omega}_0 = \omega_0/\kappa = 10.0$, $\tilde{\omega}_b = \omega_b/\kappa = 10.0$. The temperature parameter is given by $\tilde{T} = k_B T/\hbar\kappa = 1.0$ (dashed line) and $\tilde{T} = 100.0$ (solid line).

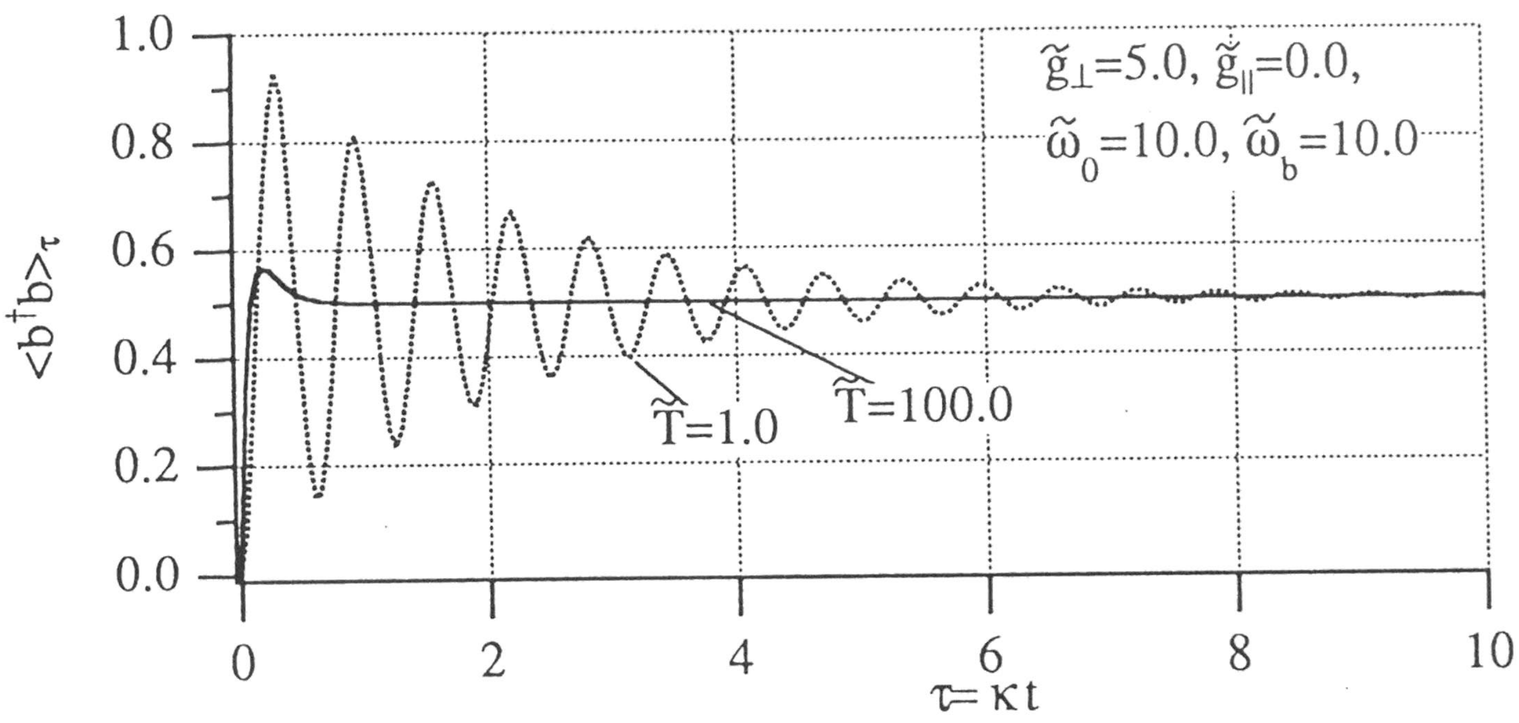
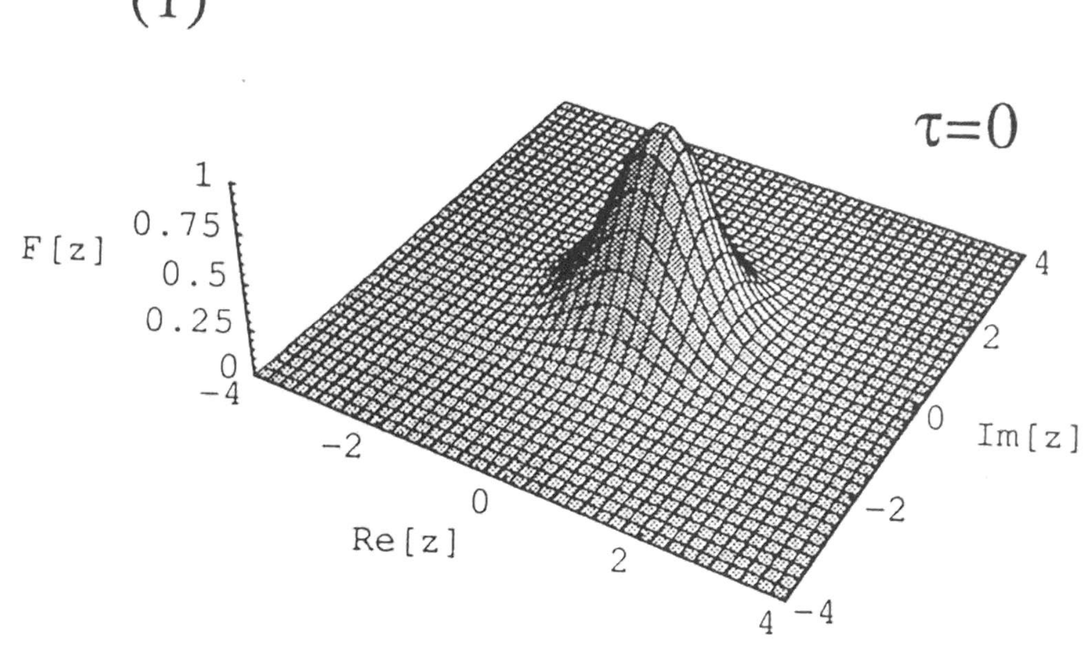


Fig.2.1-b Time evolution of $\langle b^\dagger b\rangle_\tau$.

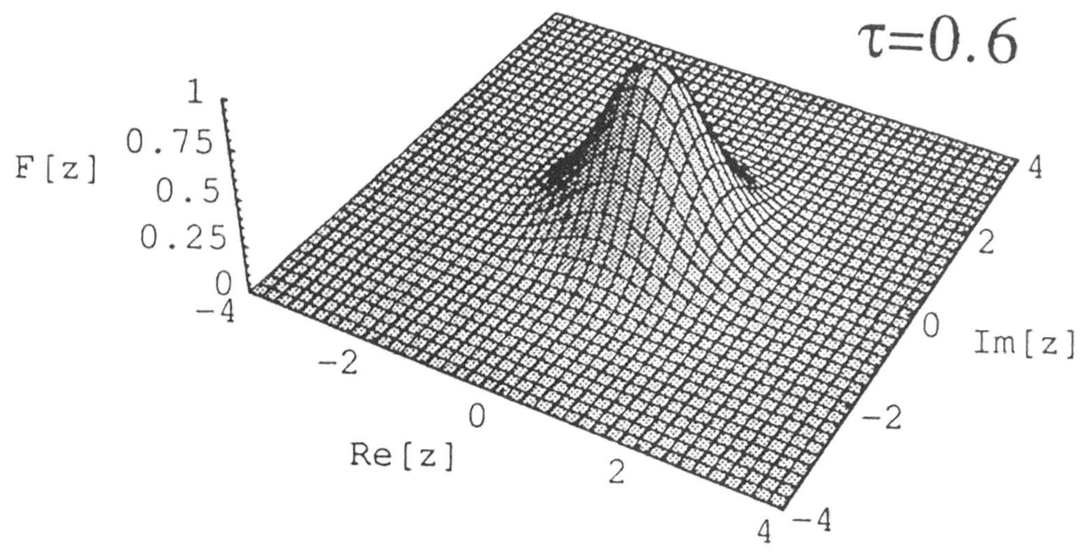
(1)



(2)

Fig.2.1-c Quasi-probability density $F(z,t)$ for the time (1) $\tau = 0$, (2) $\tau = 0.3$.

(3)



(4)

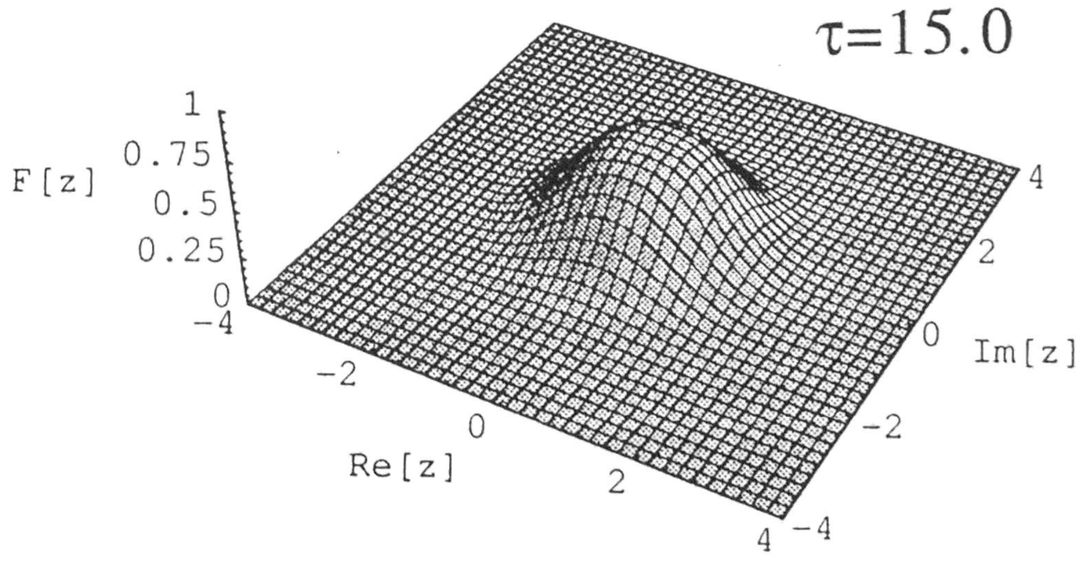


Fig.2.1-c —continued. Quasi-probability density $F(z,t)$ for the time (3) $\tau = 0.6$, (4) $\tau = 15$.

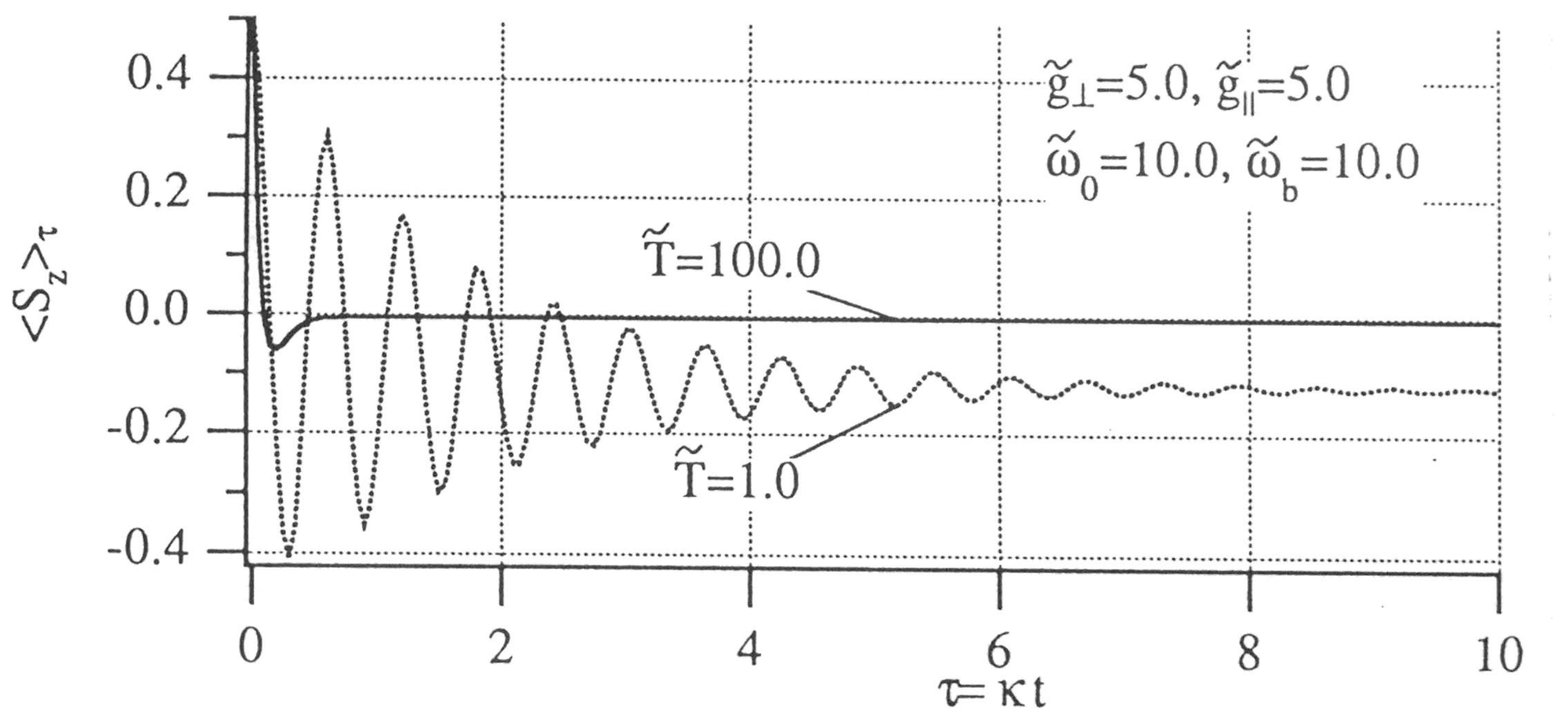


Fig.2.2 Time evolution of $\langle S_z \rangle_\tau$ for parameters $\tilde{g}_\perp = 5.0$, $\tilde{g}_\parallel = 5.0$, $\tilde{\omega}_0 = 10.0$, $\tilde{\omega}_b = 10.0$.

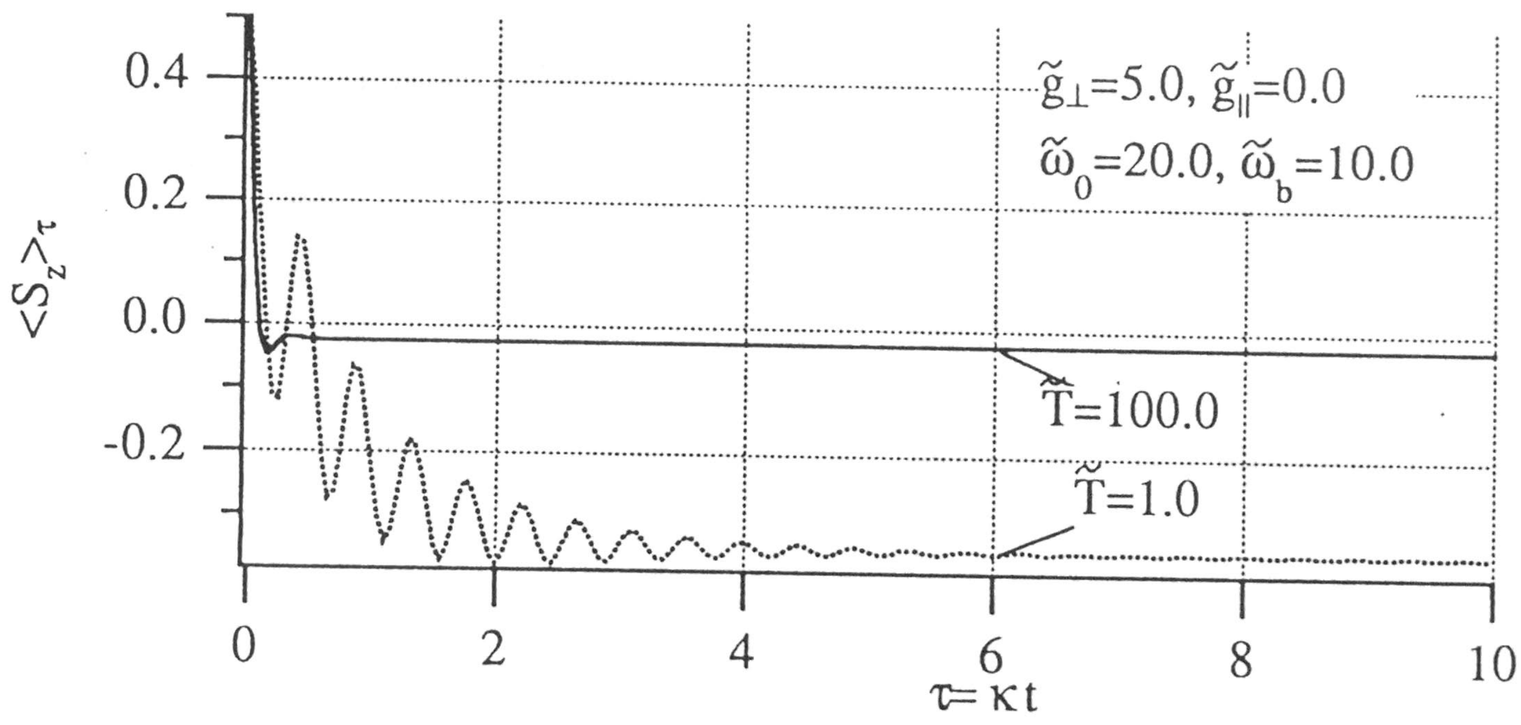


Fig.2.3 Time evolution of $\langle S_z \rangle_\tau$ for parameters $\tilde{g}_\perp = 5.0$, $\tilde{g}_{||} = 0.0$, $\tilde{\omega}_0 = 20.0$, $\tilde{\omega}_b = 10.0$.

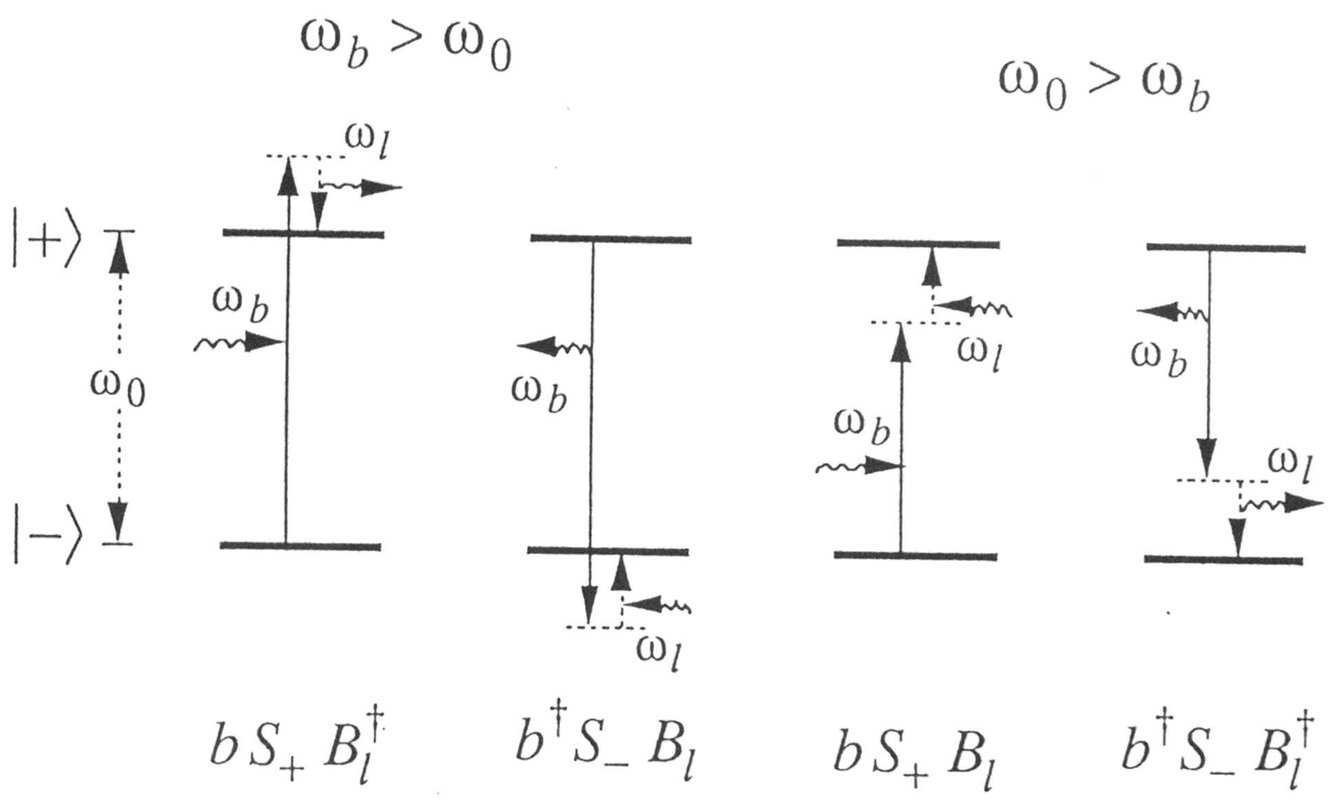


Fig.2.4 Illustration of the interaction between the relevant system and the reservoir.

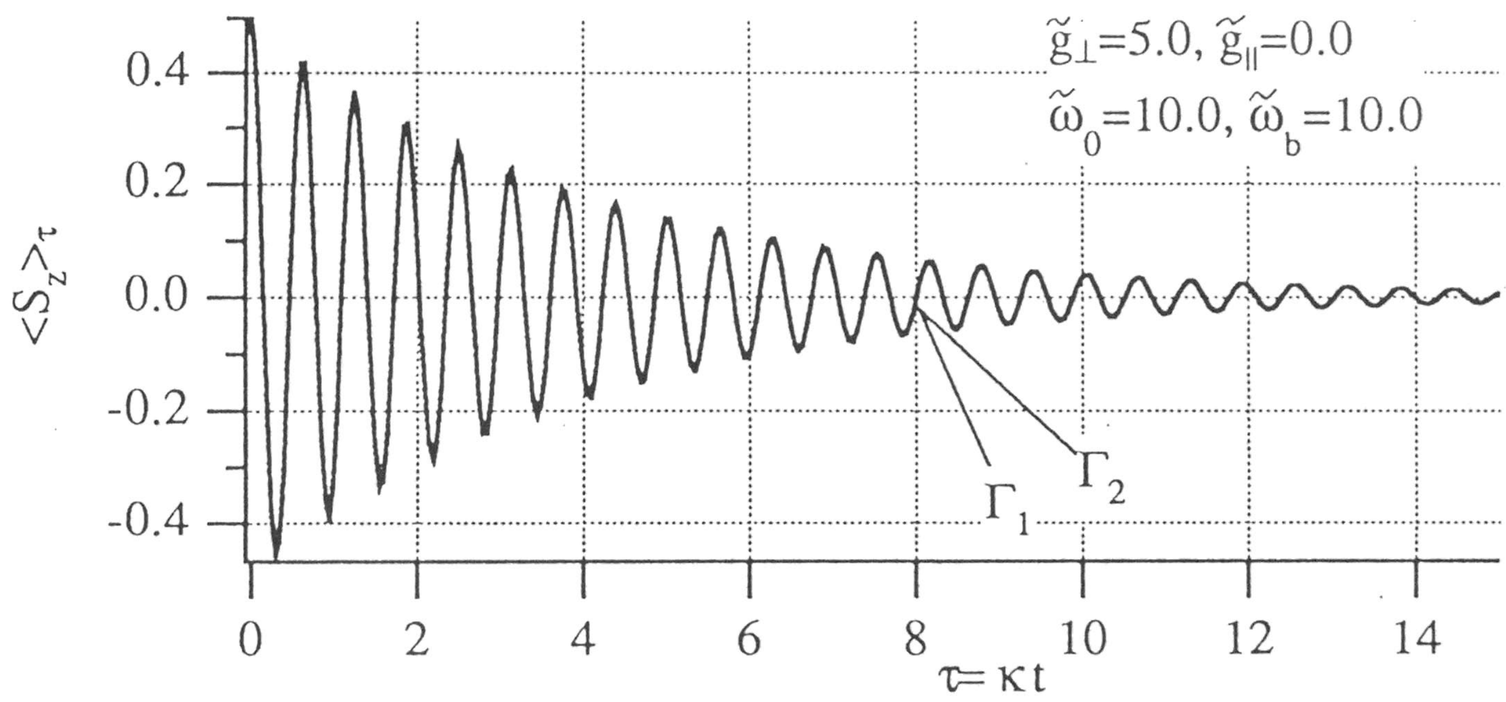


Fig.2.5-a Time evolution of $\langle S_z \rangle_\tau$ by damping operators Γ_1 and Γ_2. Time evolution by the two damping operators coincides. The system parameters are the same as in Figs.2.1 and $\tilde{T} = 1.0$.

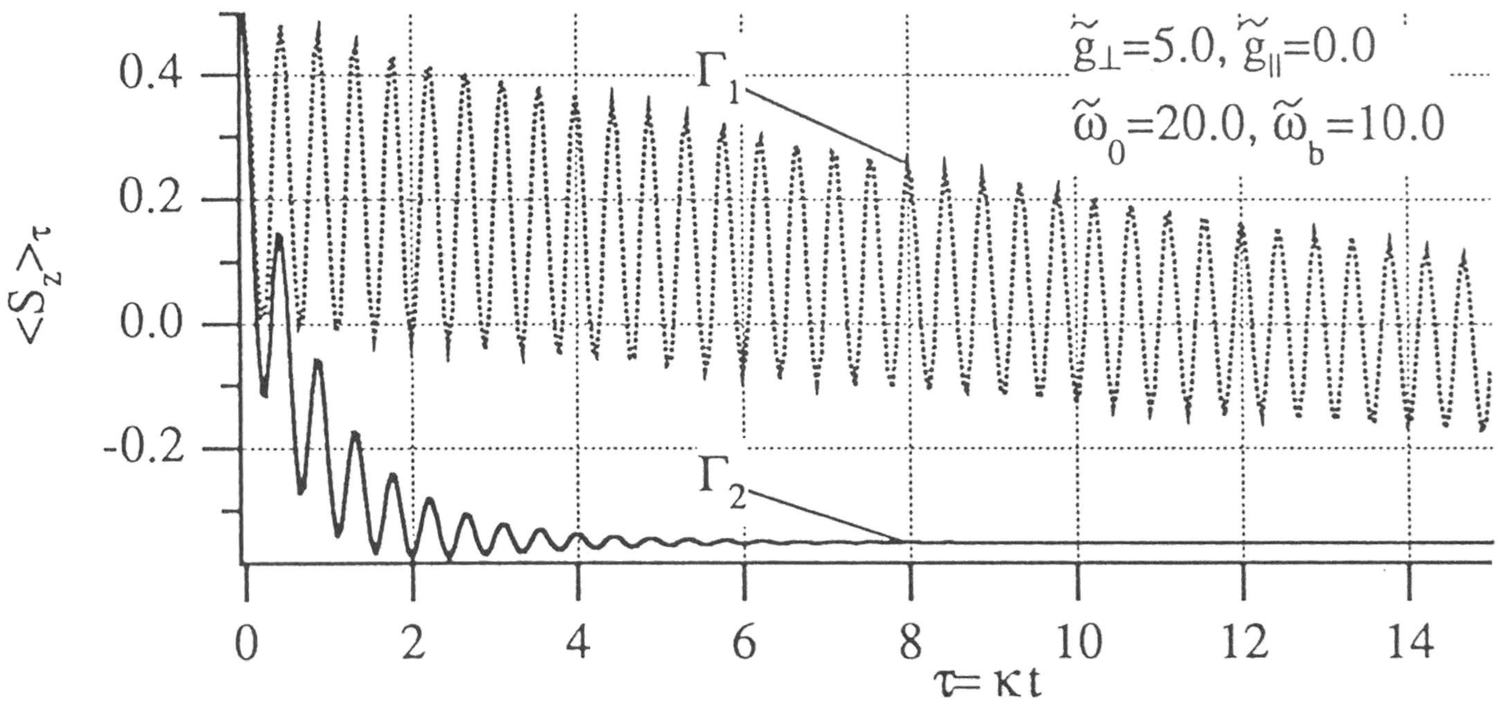


Fig.2.5-b Time evolution of $\langle S_z \rangle_\tau$ by damping operator Γ_1 (dashed line) and Γ_2 (solid line). The system parameters are the same as in Fig.2.3 and $\tilde{T} = 1.0$.

2.7 Short summary and discussions

In the present chapter, we have introduced a solvable relaxation model which determines time evolution of the Jaynes-Cummings model. Interaction between the relevant system and the reservoir has a form to modulate the coupling constant $g_\perp$ within the relevant system. After obtaining a quantal master equation for the density matrix $\rho(t)$ by the method of TCL formalism, we expand the quantal master equation in terms of the eigenstates of H to give a set of simple and tractable equations (2.4.22)-(2.4.25). Some of the expectation values are simply written in terms of the components of the density matrix. With the initial condition given in section 2.6, we studied quantum dynamics of the coupled relaxation system including time evolution of the quasi-probability density of the boson system.

It should be stressed that our theory is valid irrespective of the strength of the coupling constant $g_\perp$ between the relevant subsystems. That is, we can treat the case of strong interaction between the subsystems S and b [See (2.2.4)]. This is beyond applicability of the usual perturbation theory and the conventional master equation approach.

In our model, the basic equations are closed in the same subspace specified by n or (n, m) as is seen from (2.4.22)-(2.4.25). The relaxation mechanism works only in the subspace. In other words, there is no "jump" or spontaneous emission between the different subspaces. This is entirely due to the property of the interaction (2.2.7). The initial condition can be realized when, for instance, the temperature is very low and/or the relevant system is artificially pumped into a special set of states. Thus, there is a possibility to control the spontaneous emission when our mechanism of relaxation is realized. On the contrary, the coupling (2.4.26) gives an equation of motion in the tridiagonal form with respect to the index (n, m) [8]. This coupling implies mixing among the subspaces.

Appendix

2.A Damping operator

In a framework of the damping theory [6], we can eliminate irrelevant variable **B** systematically. Especially, for the interaction Hamiltonian

$$H_{AB} = \hbar g_B \mathbf{A} \cdot \mathbf{B} \tag{2.A.1}$$

where **A** is the relevant variable, we have the equation of motion for the density matrix of the relevant system

$$\frac{\partial}{\partial t}\hat{\rho}\left(t\right) = -g_B^2 \int_0^t dt' \left\langle \left[\hat{\mathbf{A}}(t)\cdot\hat{\mathbf{B}}(t)\right]^{\times}\left[\hat{\mathbf{A}}(t')\cdot\hat{\mathbf{B}}(t')\right]^{\times}\right\rangle_B \hat{\rho}\left(t\right) \tag{2.A.2}$$

when the interaction is weak. We have introduced a symbol $C^{\times}D \equiv [C, D]$. We have also introduced the following quantities:

$$\hat{\rho}\left(t\right) = \mathrm{e}^{iH_A t/\hbar}\rho(0)\mathrm{e}^{-iH_A t/\hbar}, \tag{2.A.3}$$

$$\hat{\mathbf{A}}\left(t\right) = \mathrm{e}^{iH_A t/\hbar}\mathbf{A}\mathrm{e}^{-iH_A t/\hbar}, \tag{2.A.4}$$

$$\hat{\mathbf{B}}\left(t\right) = \mathrm{e}^{iH_B t/\hbar}\mathbf{B}\mathrm{e}^{-iH_B t/\hbar}, \tag{2.A.5}$$

where $\rho(t)$ represents the reduced density matrix of the relevant system and H_A and H_B are the Hamiltonians of the relevant system and the irrelevant system (reservoir), respectively. The average symbol $\langle\cdots\rangle$ is defined by $\langle\cdots\rangle = \mathrm{tr}_B\left(\cdots\rho_B\right)$ where ρ_B is the reservoir density matrix. We have assumed $\langle\mathbf{B}\rangle_B = 0$ and imposed initial condition that $W(0) = \rho(0)\cdot\rho_B$.

The equation (2.A.2) leads to the following quantal master equation in the Schrödinger picture:

$$\frac{\partial}{\partial t}\rho\left(t\right) = \frac{1}{i\,\hbar}\left[H_A\ ,\rho\left(t\right)\right] + \mathit{\Gamma}\ \rho\left(t\right) \tag{2.A.6}$$

where

$$\begin{aligned}\mathit{\Gamma}\ \rho\left(t\right) &= g_B^2\int_0^t dt' \sum_{j,t}\left\{\left\langle\hat{B}_j\left(t'\right)\hat{B}_l\right\rangle_B\left[\hat{A}_l(-t')\rho(t), A_j\right]\right. \\ &+ \left.\left\langle\hat{B}_l\left(-t'\right)\hat{B}_j\right\rangle_B\left[A_j, \rho(t)\hat{A}_l(-t')\right]\right\}. \end{aligned} \tag{2.A.7}$$

2.B Coefficient Matrices

The coefficient matrices of the coherent motion in (2.4.22)-(2.4.25) are given by

$$\mathbf{L}^{nm} = \frac{1}{\hbar}\begin{pmatrix} E_+^n - E_+^m & 0 & 0 & 0 \\ 0 & E_+^n - E_-^m & 0 & 0 \\ 0 & 0 & E_-^n - E_+^m & 0 \\ 0 & 0 & 0 & E_-^n - E_-^m \end{pmatrix}, \tag{2.B.1}$$

$$\mathbf{L}^{n\ -1} = \frac{1}{\hbar}\begin{pmatrix} E_+^n - E_0 & 0 \\ 0 & E_-^n - E_0 \end{pmatrix}, \tag{2.B.2}$$

and

$$\mathbf{L}^{-1\ m} = \frac{1}{\hbar}\begin{pmatrix} E_0 - E_+^m & 0 \\ 0 & E_0 - E_-^m \end{pmatrix} \tag{2.B.3}$$

together with 2.3.8 and 2.3.9.

The coefficient matrices of the damping term in (2.4.22) are given by

$$\Gamma(n,m) = \begin{pmatrix} \gamma_1(n,m) & \gamma_2(m,n) & \gamma_2(n,m) & \gamma_3(n,m) \\ \gamma_4(m,n) & \gamma_5(n,m) & \gamma_6(n,m) & \gamma_7(n,m) \\ \gamma_4(n,m) & \gamma_6(m,n) & \gamma_5(m,n) & \gamma_7(m,n) \\ \gamma_8(n,m) & \gamma_9(n,m) & \gamma_9(m,n) & \gamma_{10}(n,m) \end{pmatrix} \tag{2.B.4}$$

where

$$\begin{aligned} \gamma_1(n,m) &= -(n+1)(\tilde{n}(n)+1)\left(\sin^4\theta_n + \cos^4\theta_n\right) \\ &- (m+1)(\tilde{n}(m)+1)\left(\sin^4\theta_m + \cos^4\theta_m\right), \qquad (2.B.5) \\ \gamma_2(n,m) &= -\sqrt{n+1}\tilde{n}(n)\cos 2\theta_n \xi, \qquad (2.B.6) \\ \gamma_3(n,m) &= \sqrt{n+1}\sqrt{m+1}\,(\tilde{n}(n)+\tilde{n}(m)) \\ &\times \left\{\sin^2\theta_n \sin^2\theta_m + \cos^2\theta_n \cos^2\theta_m\right\}, \qquad (2.B.7) \\ \gamma_4(n,m) &= \sqrt{n+1}\,(\tilde{n}(n)+1)\cos 2\theta_n \eta, \qquad (2.B.8) \\ \gamma_5(n,m) &= -(n+1)(\tilde{n}(n)+1)\left(\sin^4\theta_n + \cos^4\theta_n\right) \\ &- (m+1)\,\tilde{n}(m)\left(\sin^4\theta_m + \cos^4\theta_m\right), \qquad (2.B.9) \end{aligned}$$

$$
\begin{aligned}
\gamma_6(n,m) &= \sqrt{n+1}\sqrt{m+1}\,(\tilde{n}(n)+\tilde{n}(m)+1) \\
&\times \left\{\sin^2\theta_n\cos^2\theta_m+\sin^2\theta_n\cos^2\theta_m\right\}, \qquad (2.B.10)\\
\gamma_7(n,m) &= \sqrt{n+1}\tilde{n}(n)\cos 2\theta_n\eta, \qquad (2.B.11)\\
\gamma_8(n,m) &= \sqrt{n+1}\sqrt{m+1}\,(\tilde{n}(n)+\tilde{n}(m)+1) \\
&\times \left\{\sin^2\theta_n\sin^2\theta_m+\cos^2\theta_n\cos^2\theta_m\right\}, \qquad (2.B.12)\\
\gamma_9(n,m) &= \sqrt{n+1}\,(\tilde{n}(n)+1)\cos 2\theta_n\xi, \qquad (2.B.13)\\
\gamma_{10}(n,m) &= -(n+1)\,\tilde{n}(n)\left(\sin^4\theta_n+\cos^4\theta_n\right) \\
&- (m+1)\,\tilde{n}(m)\left(\sin^4\theta_m+\cos^4\theta_m\right) \qquad (2.B.14)
\end{aligned}
$$

with

$$
\begin{aligned}
\xi &= \sqrt{n+1}\sin\theta_n\cos\theta_n-\sqrt{m+1}\sin\theta_m\cos\theta_m, \qquad (2.B.15)\\
\eta &= \sqrt{n+1}\sin\theta_n\cos\theta_n+\sqrt{m+1}\sin\theta_m\cos\theta_m. \qquad (2.B.16)
\end{aligned}
$$

Finally the coefficient matrices in (2.4.23) and (2.4.24) are given by

$$
\Gamma(n) = (n+1)\begin{pmatrix}\gamma_1(n) & \gamma_2(n)\\ \gamma_3(n) & \gamma_4(n)\end{pmatrix}. \qquad (2.B.17)
$$

where

$$
\begin{aligned}
\gamma_1(n) &= -(\tilde{n}(n)+1)\left(\sin^4\theta_n+\cos^4\theta_n\right), \qquad (2.B.18)\\
\gamma_2(n) &= -\tilde{n}(n)\cos 2\theta_n\sin\theta_n\cos\theta_n, \qquad (2.B.19)\\
\gamma_3(n) &= (\tilde{n}(n)+1)\cos 2\theta_n\sin\theta_n\cos\theta_n, \qquad (2.B.20)\\
\gamma_4(n) &= -\tilde{n}(n)\left(\sin^4\theta_n+\cos^4\theta_n\right). \qquad (2.B.21)
\end{aligned}
$$

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