Chapter 3

A Relaxation Model for the Dissipative Jaynes-Cummings Model toward Thermal Equilibrium

With use of the Jaynes-Cummings model with relaxation mechanisms, quantum dynamics of a strongly coupled dissipative system toward thermal equilibrium is investigated. Our relaxation model ensures the coupled system evolving in time to the correct canonical distribution in thermal equilibrium. The quantal master equation is expanded in terms of the eigenstates of the whole coupled system. Time evolution of elements of the reduced density matrix are described by the vector tri-diagonal differential equation. The relaxation process reveals itself through the dynamics of these elements resulting in the canonical distribution. Quantum characteristics are found both in the short time regime and the long time regime. The short time regime is characterized by decoherence process, which represents the phase relaxation, whereas the long time relaxation process is dominated by the diagonal process of the energy relaxation.

3.1 Introduction

The Jaynes-Cummings model [1] is one of the exactly solvable quantum coupled systems. This is a model of a two-level atom (or a molecule) interacting with a single mode boson field near the resonance. There are a lot of works on this model [2] in the past decades and recently it has been received a renewed attention in connection with micromaser [3] and cavity QED [4].

Real systems are usually surrounded by their respective environments. Then the dissipative Jaynes-Cummings model was studied by the generalized master equation approach [5, 6]. To take into account the effect of dissipation, usual treatment [2] couples with the two independent constituent dissipative systems, namely, a dissipative two level atom and a boson field. However that kind of treatment can not give the correct canonical distribution of the whole coupled system but the thermal equilibrium of each subsystem only. Especially for the strong coupling between the two subsystems and for the long time behavior, this approximation is insufficient. This point was also noted by Cresser [8] and he proposed a modified master equation which ensures the thermal equilibrium of the coupled system. He investigated the properties of the master equation in the long time limit and for the case of $T = 0$ (T the temperature).

We have made a theoretical framework of the dissipative Jaynes-Cummings model, which ensures the canonical distribution of the coupled system at finite temperature T irrespective of the interaction strength between the atom and the boson field [9]. We have obtained the quantal master equation for the reduced density operator of the coupled system and solved the equation rigorously. It is possible to treat the modulation effect of the coupling between the subsystems due to the reservoir as has been shown in the previous chapter. This effect can not be taken into account by the existing theories.

With this formulation, we investigate quantum dynamics toward the thermal equilibrium by observing time-evolution of matrix elements of the reduced density matrix, which is represented by the vector tridiagonal differential equation. It should be emphasized that the transient dynamics both in the short time regime and the long

time regime can be obtained systematically by our method. We study this dissipative Jaynes-Cummings model as a prototype of solvable quantum coupled relaxation system.

Further, this formulation is valid even if the coupling constant between the subsystems is very strong. In this regime, the coupling is too strong to regard the model as the one derived from the ordinally atom-field interaction model under the rotating wave approximation. Instead, we propose the model in a broad sense, namely, a model for a spin 1/2 particle interacting with a boson system. We also investigate the relaxation process of the strong coupling case where reversal of the energy level occurs.

In this chapter, our system Hamiltonian is given in section 2. In section 3, we introduce a unified representation of the master equations for the several types of interactions with a reservoir. We derive and rewrite the basic equation from the unified master equation view point in section 4. All the coefficient matrices in the basic equation are given in Appendix. In section 5, a consideration on the physical picture of the basic equation is given. In section 6 we impose initial conditions and we show the dynamics of the elements of the diagonal part in the boson quantum number of the reduced density matrix.

3.2 System

Our relevant system is composed of a slightly extended version of the Jaynes-Cummings model in the rotating wave approximation represented by the Hamiltonian [9]

$$H = H_S + H_b + H_i \tag{3.2.1}$$

where

$$H_S = \hbar\omega_0 S_z, \tag{3.2.2}$$

$$H_b = \hbar\omega_b b^\dagger b \tag{3.2.3}$$

and

$$H_i = \hbar g_{||} b^\dagger b S_z + \hbar g_\perp \left(b S_+ + b^\dagger S_- \right) \tag{3.2.4}$$

$\boldsymbol{S} = (S_x,\ S_y,\ S_z)$ being the spin 1/2 operator with $S_\pm = S_x \pm i S_y$, the Larmor (angular) frequency ω_0, and the annihilation (creation) operator b $\left(b^\dagger\right)$ having the (angular) frequency ω_b for the bosons. The first term on the right hand side of (3.2.4) which does not appear in the ordinal Jaynes-Cummings model represents the adiabatic interaction. We impose a condition $g_{||} < 2\omega_b$ for the adiabatic coupling constant. This condition is required for the convergence of the density matrix at equilibrium state.

This system is exactly solvable irrespective of the strength of the coupling constant $g_\perp$. The system Hamiltonian has eigenstates [6, 9]

$$|\varphi(n,1)\rangle = \cos\theta_n |n+1,-1\rangle + \sin\theta_n |n,+1\rangle\,, \tag{3.2.5}$$

$$|\varphi(n,2)\rangle = -\sin\theta_n |n+1,-1\rangle + \cos\theta_n |n,+1\rangle\,, \tag{3.2.6}$$

and

$$|0,-1\rangle\,. \tag{3.2.7}$$

In these expressions $|n,+1\rangle$ represents the state with n bosons and spin-up whereas $|n,-1\rangle$ spin-down. The states (3.2.5) and (3.2.6) are sometimes called "dressed states" [10, 11] where the eigenstates are labeled by a set of quantum numbers, (n,α). We see that n is a quantum number of the boson subsystem origin and $\alpha = 1,2$ is a quantum

number of the spin subsystem origin. We call in this paper n "a boson quantum number" and α "a spin quantum number".

The eigenvalues of H, that is, solution of the following equations

$$H \ |\varphi(n,1)\rangle = E_+^n \ |\varphi(n,1)\rangle, \tag{3.2.8}$$

$$H \ |\varphi(n,2)\rangle = E_-^n \ |\varphi(n,2)\rangle, \tag{3.2.9}$$

$$H \ |0,-1\rangle = E_0 \ |0,-1\rangle, \tag{3.2.10}$$

is given by

$$E_\pm^n = \hbar\left\{\omega_b\left(n+\frac{1}{2}\right) - \frac{g_{||}}{4} \pm \lambda(n)\right\}, \tag{3.2.11}$$

and

$$E_0 = -\frac{\hbar\omega_0}{2} \tag{3.2.12}$$

with

$$\lambda(n) = \sqrt{\frac{1}{4}\left(\Delta\omega - \frac{g_{||}(2n+1)}{2}\right)^2 + g_\perp^2(n+1)} \tag{3.2.13}$$

where the detuning $\Delta\omega$ being defined by $\Delta\omega = \omega_b - \omega_0$. The effect of the adiabatic interaction is to give an additional shift on $\Delta\omega$. The angle θ_n is determined to give the following relation:

$$\tan\theta_n = \frac{g_\perp\sqrt{n+1}}{\frac{1}{2}\left(\Delta\omega - \frac{g_{||}(2n+1)}{2}\right) + \lambda(n)} \tag{3.2.14}$$

for $g_\perp \neq 0$.

We see from (3.2.11), even if the near resonance condition is satisfied, that is, for $\Delta\omega << 0$, the energy levels of two eigenstates belonging to the same boson quantum number split due to the coupling represented by $g_{||}$ and $g_\perp$. For certain values of $g_{||}$ and $g_\perp$, reversal of the energy levels occurs, namely, we find $E_-^{n+1} < E_+^n$ for larger n. For instance, when $g_\perp > \sqrt{\omega_0\left(\omega_b - \frac{g_{||}}{2}\right)}$, the state $|0,-1\rangle$ is no longer a ground state although $\omega_0 > 0$. The new ground state is $|\varphi(0,2)\rangle$, which is a superposition of the states $|0,+1\rangle$ and $|1,-1\rangle$.

3.3 Quantal master equation

We consider our system (3.2.1) is surrounded by the environment that causes fluctuation and dissipation to the system. We regard our whole system is coupling with a reservoir. The reservoir consists of harmonic oscillators in thermal equilibrium at temperature T and has the canonical distribution. The Hamiltonian of the reservoir is given by

$$H_B = \hbar \sum_l \omega_l B_l^\dagger B_l \tag{3.3.1}$$

with $\omega_l > 0$ where $B_l^\dagger$ and B_l represent the reservoir variables.

The interaction Hamiltonian between the system and the reservoir is given by [9]

$$H_{SB} = \hbar \left(S_- + S_+ \right) g_{SB} \sum_l \left(B_l^\dagger + B_l \right) \tag{3.3.2}$$

and

$$H_{bB} = \hbar \left(b + b^\dagger \right) g_{bB} \sum_l \left(B_l^\dagger + B_l \right) \tag{3.3.3}$$

which describe energy dissipation from the spin 1/2 system and the boson system, respectively. In the third case, the non-adiabatic coupling constant $g_\perp$ in (3.2.4) is considered to be modified due to an interaction with a reservoir, that is, the system is perturbed by the reservoir through the Hamiltonian of the form

$$H_{iB} = \hbar \left(b S_+ + b^\dagger S_- \right) g_{iB} \sum_l \left(B_l^\dagger + B_l \right). \tag{3.3.4}$$

The reduced density matrix of the relevant system is given by

$$\rho \left(t \right) = \mathrm{tr}_B W \left(t \right) \tag{3.3.5}$$

where $W \left(t \right)$ is the density matrix of the whole system including the reservoir. The symbol tr_B stands for the trace operation over the reservoir variables. With the use of the time convolutionless (TCL) formalism [12], the quantal master equation of the relevant system is given by

$$\frac{\partial}{\partial t} \rho \left(t \right) = \frac{1}{i \hbar} \left[H \ , \rho \left(t \right) \right] + \Gamma_S \ \rho \left(t \right) + \Gamma_b \ \rho \left(t \right) + \Gamma_i \ \rho \left(t \right) \tag{3.3.6}$$

where

$$
\begin{aligned}
\Gamma_S\ \rho(t) &= \sum_l g_S^2 \int_0^t dt' \Big\{ \Big(\langle B_l^\dagger(t')\, B_l \rangle_B + \langle B_l(t')\, B_l^\dagger \rangle_B \Big) \\
&\times \ \left([S_+(-t')\, \rho(t), S_-] + [S_-(-t')\, \rho(t), S_+] \right) \\
&+ \ \Big(\langle B_l^\dagger(-t')\, B_l \rangle_B + \langle B_l(-t')\, B_l^\dagger \rangle_B \Big) \\
&\times \ \left([S_+,\ \rho(t)\, S_-(-t')] + [S_-,\ \rho(t)\, S_+(-t')] \right) \Big\}, \qquad (3.3.7)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_b\ \rho(t) &= \sum_l g_b^2 \int_0^t dt' \Big\{ \Big(\langle B_l^\dagger(t')\, B_l \rangle_B + \langle B_l(t')\, B_l^\dagger \rangle_B \Big) \\
&\times \ \Big(\left[b^\dagger(-t')\, \rho(t), b \right] + \left[b(-t')\, \rho(t), b^\dagger \right] \Big) \\
&+ \ \Big(\langle B_l^\dagger(-t')\, B_l \rangle_B + \langle B_l(-t')\, B_l^\dagger \rangle_B \Big) \\
&\times \ \Big(\left[b^\dagger,\ \rho(t)\, b(-t') \right] + \left[b,\ \rho(t)\, b^\dagger(-t') \right] \Big) \Big\}, \qquad (3.3.8)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_i\ \rho(t) &= \sum_l g_i^2 \int_0^t dt' \Big\{ \Big(\langle B_l^\dagger(t')\, B_l \rangle_B + \langle B_l(t')\, B_l^\dagger \rangle_B \Big) \\
&\times \ \Big(\left[\underline{bS_+}(-t')\, \rho(t),\ \underline{b^\dagger S_-} \right] + \left[\underline{b^\dagger S_-}(-t')\, \rho(t),\ \underline{bS_+} \right] \Big) \\
&+ \ \Big(\langle B_l^\dagger(-t')\, B_l \rangle_B + \langle B_l(-t')\, B_l^\dagger \rangle_B \Big) \\
&\times \ \Big(\left[\underline{bS_+},\ \rho(t)\ \underline{b^\dagger S_-}(-t') \right] + \left[\underline{b^\dagger S_-},\ \rho(t)\ \underline{bS_+}(-t') \right] \Big) \Big\}. \qquad (3.3.9)
\end{aligned}
$$

In the damping terms (3.3.7)-(3.3.9), time evolution of reservoir variable is determined by (3.3.1) and time evolution of the system operators $S_+(t)$, $S_-(t)$, $b(t)$, $b^\dagger(t)$, $\underline{bS_+}(t)$ and $\underline{b^\dagger S_-}(t)$ is determined by the total Hamiltonian H of the relevant system, (3.2.1). This is a necessary factor ensuring the system relaxation to the thermal equilibrium to construct the canonical distribution even when the coupling constant $g_\perp$ becomes large. But this is not an enough factor as we will see in the next sections.

3.4 Basic equation

We expand the quantal master equation (3.3.6) in terms of the eigenstates (3.2.5)-(3.2.7). We denote elements of the reduced density matrix $\rho(t)$ as

$$\rho_{\alpha\beta}^{nm}(t) = \langle \varphi(n,\alpha)| \rho(t) |\varphi(m,\beta)\rangle , \tag{3.4.1}$$

$$\rho_{0\beta}^{m}(t) = \langle 0,-1| \rho(t) |\varphi(m,\beta)\rangle , \tag{3.4.2}$$

$$\rho_{\alpha 0}^{n}(t) = \langle \varphi(n,\alpha)| \rho(t) |0,-1\rangle , \tag{3.4.3}$$

$$\rho_{00}(t) = \langle 0,-1| \rho(t) |0,-1\rangle \tag{3.4.4}$$

with $\alpha,\ \beta = 1,2$. And for later convenience, we also define the following vectors specified by a set of boson quantum numbers (n,m):

$$\boldsymbol{\rho}^{nm}(t) \equiv \begin{pmatrix} \rho_{11}^{nm}(t) \\ \rho_{12}^{nm}(t) \\ \rho_{21}^{nm}(t) \\ \rho_{22}^{nm}(t) \end{pmatrix} , \tag{3.4.5}$$

for $n,m \geq 0$, and

$$\boldsymbol{\rho}^{-1\ m}(t) \equiv \begin{pmatrix} \rho_{01}^{m}(t) \\ \rho_{02}^{m}(t) \end{pmatrix} , \tag{3.4.6}$$

$$\boldsymbol{\rho}^{n\ -1}(t) \equiv \begin{pmatrix} \rho_{10}^{n}(t) \\ \rho_{20}^{n}(t) \end{pmatrix} \tag{3.4.7}$$

and

$$\boldsymbol{\rho}^{-1\ -1}(t) \equiv \rho_{00}(t) . \tag{3.4.8}$$

By introducing these vectors, we have thus decomposed the Liouville space of the reduced density matrix into subspaces specified only by a set of boson quantum numbers.

Next, we expand the following system operators in term of the eigenbras and kets:

$$\begin{aligned} S_+(t) &= \sum_{n=0}\sum_{\alpha,\ \beta} \mathrm{e}^{i\epsilon_{\alpha\beta}^{n} t/\hbar}\ f_{\alpha\beta}(n)\, |\varphi(n+1,\alpha)\rangle \langle \varphi(n,\beta)| \\ &+ \sum_{\alpha} \mathrm{e}^{i\epsilon_{\alpha 0} t/\hbar} f_{\alpha 0}\, |\varphi(0,\alpha)\rangle \langle 0,-1|, \end{aligned} \tag{3.4.9}$$

$$b^{\dagger}(t) = \sum_{n=0}\sum_{\alpha,\ \beta} e^{i\epsilon^{n}_{\alpha\beta}t/\hbar}\ d_{\alpha\beta}(n)\,|\varphi(n+1,\alpha)\rangle\,\langle\varphi(n,\beta)| + \sum_{\alpha} e^{i\epsilon_{\alpha 0}t/\hbar} d_{\alpha 0}\,|\varphi(0,\alpha)\rangle\,\langle 0,-1|, \tag{3.4.10}$$

and

$$\underline{bS_{+}}(t) = \sum_{n=0}\sum_{\alpha,\ \beta} e^{i\delta^{n}_{\alpha\beta}t/\hbar}\ h_{\alpha\beta}(n)\,|\varphi(n,\alpha)\rangle\,\langle\varphi(n,\beta)| \tag{3.4.11}$$

where

$$f_{11}(n) = \cos\theta_n \sin\theta_{n+1}, \tag{3.4.12}$$

$$f_{12}(n) = -\sin\theta_n \sin\theta_{n+1}, \tag{3.4.13}$$

$$f_{21}(n) = \cos\theta_n \cos\theta_{n+1}, \tag{3.4.14}$$

$$f_{22}(n) = -\sin\theta_n \cos\theta_{n+1}, \tag{3.4.15}$$

$$f_{10} = \sin\theta_0, \tag{3.4.16}$$

$$f_{20} = \cos\theta_0, \tag{3.4.17}$$

$$d_{11}(n) = \sqrt{n+2}\cos\theta_n \cos\theta_{n+1} + \sqrt{n+1}\sin\theta_n \sin\theta_{n+1}, \tag{3.4.18}$$

$$d_{12}(n) = -\sqrt{n+2}\sin\theta_n \cos\theta_{n+1} + \sqrt{n+1}\cos\theta_n \sin\theta_{n+1}, \tag{3.4.19}$$

$$d_{21}(n) = -\sqrt{n+2}\cos\theta_n \sin\theta_{n+1} + \sqrt{n+1}\sin\theta_n \cos\theta_{n+1}, \tag{3.4.20}$$

$$d_{22}(n) = \sqrt{n+2}\sin\theta_n \sin\theta_{n+1} + \sqrt{n+1}\cos\theta_n \cos\theta_{n+1}, \tag{3.4.21}$$

$$d_{10} = \cos\theta_0, \tag{3.4.22}$$

$$d_{20} = -\sin\theta_0, \tag{3.4.23}$$

$$h_{11}(n) = \sqrt{n+1}\sin\theta_n \cos\theta_n, \tag{3.4.24}$$

$$h_{12}(n) = -\sqrt{n+1}\sin^2\theta_n, \tag{3.4.25}$$

$$h_{21}(n) = \sqrt{n+1}\cos^2\theta_n, \tag{3.4.26}$$

$$h_{22}(n) = -h_{11}(n). \tag{3.4.27}$$

These are written in terms of the quantities represented by the energy difference between the neighboring boson quantum number states:

$$\epsilon^{n}_{11} \equiv E^{n+1}_{+} - E^{n}_{+}, \tag{3.4.28}$$

$$\epsilon_{12}^{n} \equiv E_{+}^{n+1} - E_{-}^{n}, \tag{3.4.29}$$

$$\epsilon_{21}^{n} \equiv E_{-}^{n+1} - E_{+}^{n}, \tag{3.4.30}$$

$$\epsilon_{22}^{n} \equiv E_{-}^{n+1} - E_{-}^{n}, \tag{3.4.31}$$

$$\epsilon_{10} \equiv E_{+}^{0} - E_{0}, \tag{3.4.32}$$

$$\epsilon_{20} \equiv E_{-}^{0} - E_{0}, \tag{3.4.33}$$

and the quantities represented by the energy difference between the two states with the same boson quantum number:

$$\delta_{11}^{n} \equiv 0, \tag{3.4.34}$$

$$\delta_{12}^{n} \equiv 2\lambda(n), \tag{3.4.35}$$

$$\delta_{21}^{n} \equiv -2\lambda(n), \tag{3.4.36}$$

$$\delta_{22}^{n} \equiv 0. \tag{3.4.37}$$

We also have $S_{-}(t) = \{S_{+}(t)\}^{\dagger}$, $b(t) = \left\{b^{\dagger}(t)\right\}^{\dagger}$ and $\underline{b^{\dagger}S_{-}}(t) = \left\{\underline{bS_{+}}(t)\right\}^{\dagger}$. It must be pointed out that the operators $S_{+}(t)$ and $b^{\dagger}(t)$ increase the boson quantum numbers of eigenstates by one when they operate on the eigenstates, whereas the operators $S_{-}(t)$ and $b(t)$ decrease by one and the operators $\underline{bS_{+}}(t)$ and $\underline{b^{\dagger}S_{-}}(t)$ do not change the boson quantum numbers. We can expand all the terms consisting of the system operators (3.3.7)-(3.3.9) in terms of the eigenstates employing the above character of the operators.

We further assume that the correlation time of the reservoir is much shorter than the relaxation time of the system confining ourselves to the narrowing limit. Moreover, we take into account only the real part of the correlation function of the reservoir in the damping terms (3.3.7)-(3.3.9), while the imaginary part causing frequency shift may be neglected generally [6]. In the narrowing limit, time integrals in (3.3.7)-(3.3.9) give the delta functions. Due to the condition $\omega_l > 0$, the correlation functions of the reservoir variables in (3.3.7)-(3.3.9) with the delta functions are calculated to give the functions $\hat{n}_{-}(X)$ and $\hat{n}_{+}(X)$ defined by

$$\hat{n}_{-}(X) \equiv \left(\exp\left[X/k_B T\right] - 1\right)^{-1} \equiv \hat{n}(X), \tag{3.4.38}$$

$$\hat{n}_{+}(X) \equiv -\left(\exp\left[-X/k_B T\right] - 1\right)^{-1} = \hat{n}(X) + 1, \tag{3.4.39}$$

for $X > 0$, and

$$\hat{n}_{-}(X) \equiv -\left(\exp\left[X/k_BT\right]-1\right)^{-1} = \hat{n}(-X)+1, \tag{3.4.40}$$

$$\hat{n}_{+}(X) \equiv \left(\exp\left[-X/k_BT\right]-1\right)^{-1} = \hat{n}(-X), \tag{3.4.41}$$

for $X < 0$,

$$\hat{n}_{-}(X) \equiv 0, \tag{3.4.42}$$

$$\hat{n}_{+}(X) \equiv 0 \tag{3.4.43}$$

for $X = 0$.

The function (3.4.38) is considered to be a generalization of the thermal photon number

$$\bar{n} = \left(\exp\left[\hbar\omega_b/k_BT\right]-1\right)^{-1} \tag{3.4.44}$$

which appears in the generalized master equation for a damped harmonic oscillator. For the damped harmonic oscillator, the relaxation process is characterized by the damping rate κ and $\bar{n}$, the latter of which is expressed in terms of ω_b (See (3.4.44)). This is because of the above mentioned delta function originating from the time integration. In other words, the relaxation process is essentially determined by the reservoir mode which is in resonant with the harmonic oscillator energy $\hbar\omega_b$. On the other hand, we have to put the energy difference (3.4.28)-(3.4.37) into X of the functions (3.4.38)-(3.4.43). Energy difference (level spacing) $\epsilon^n_{\alpha\beta}$ can take various values depending on the quantum numbers n, α and β in contrast with the harmonic oscillator whose level spacings are constant. Thus the system is coupled with many modes of reservoir. This is the most different point from the conventional treatment, where the two independently dissipative systems are coupled, so to speak, mechanically.

After certain manipulations, we have a set of vector tri-diagonal differential equations of the form:

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{\rho}^{nm}(t) &= \kappa'\,\mathbf{p}^{nm}\boldsymbol{\rho}^{n-1\ m-1}(t)\\ &- \left(i\,\mathbf{L}^{nm}+\kappa\,\mathbf{q}^{nm}\right)\boldsymbol{\rho}^{nm}(t)\\ &+ \kappa'\,\mathbf{r}^{nm}\boldsymbol{\rho}^{n+1\ m+1}(t)\end{aligned} \tag{3.4.45}$$

where we have defined

$$\kappa' \, \mathbf{p}^{nm} \equiv \kappa_S \mathbf{p}_S{}^{nm} + \kappa_b \mathbf{p}_b{}^{nm}, \tag{3.4.46}$$

$$\kappa \, \mathbf{q}^{nm} \equiv \kappa_S \mathbf{q}_S{}^{nm} + \kappa_b \mathbf{q}_b{}^{nm} + \kappa_i \mathbf{q}_i{}^{nm}, \tag{3.4.47}$$

$$\kappa' \, \mathbf{r}^{nm} \equiv \kappa_S \mathbf{r}_S{}^{nm} + \kappa_b \mathbf{r}_b{}^{nm} \tag{3.4.48}$$

for $n, m \geq -1$ with damping constants κ_S, κ_b and κ_i and the coefficient matrices $\mathbf{L}^{nm}$, $\mathbf{p}_S{}^{nm}$, $\mathbf{p}_b{}^{nm}$, $\mathbf{q}_S{}^{nm}$, $\mathbf{q}_b{}^{nm}$, $\mathbf{q}_i{}^{nm}$, $\mathbf{r}_S{}^{nm}$, $\mathbf{r}_b{}^{nm}$ which are explicitly given in Appendix. We call (3.4.45) the basic equation. We also obtain an analytic solution of (3.4.45) in the form of continued fraction by the method of Laplace transform [9, 13]:

$$\boldsymbol{\rho}^{nm}[s] = \mathbf{u}^{n-1\ m-1}[s] \, \boldsymbol{\rho}^{n-1\ m-1}[s] + \mathbf{M}^{nm}[s] \tag{3.4.49}$$

where

$$\boldsymbol{u}^{n-1\ m-1}[s] = \mathbf{N}^{nm}[s] \, \mathbf{p}^{nm}, \tag{3.4.50}$$

$$\mathbf{M}^{nm}[s] = \mathbf{N}^{nm}[s] \left(\boldsymbol{\rho}^{nm}[0] + \mathbf{r}^{nm} \mathbf{M}^{n+1\ m+1}[s] \right) \tag{3.4.51}$$

with

$$\mathbf{N}^{nm}[s] = \left(\frac{s \cdot \mathbf{I} + i\mathbf{L}^{nm} + \kappa \mathbf{q}^{nm}}{\kappa'} + \mathbf{r}^{nm} \mathbf{u}^{nm}[s] \right)^{-1} \tag{3.4.52}$$

and conditions $\mathbf{u}^{n\ -2}[s] \equiv 0$, $\mathbf{u}^{-2\ m}[s] \equiv 0$ and $\mathbf{u}^{-2\ -2}[s] \equiv 0$. In these equations, the Laplace transform of $A(t)$ is represented by $A[s]$.

Further, it is easy to extend our formulation to the system whose interaction Hamiltonian with the reservoir is given by

$$H_{bB^2} = \hbar \left(b + b^\dagger \right) g_{bB^2} \sum_l \left\{ \left(B_l^\dagger \right)^2 + (B_l)^2 \right\}. \tag{3.4.53}$$

In this case, we get the basic equation in the same form replacing the functions (3.4.38)-(3.4.43) in the coefficients matrices by the following functions

$$\hat{m}_-(X) \equiv \exp[-X/2k_BT] \, (\exp[-X/2k_BT] - 1)^{-2}, \tag{3.4.54}$$

$$\hat{m}_+(X) \equiv (\exp[-X/2k_BT] - 1)^{-2} \tag{3.4.55}$$

for $X > 0$, and

$$\hat{m}_{-}(X) \equiv (\exp[X/2k_BT] - 1)^{-2}, \tag{3.4.56}$$

$$\hat{m}_{+}(X) \equiv \exp[X/2k_BT]\ (\exp[X/2k_BT] - 1)^{-2}, \tag{3.4.57}$$

for $X < 0$,

$$\hat{m}_{-}(X) \equiv 0, \tag{3.4.58}$$

$$\hat{m}_{+}(X) \equiv 0 \tag{3.4.59}$$

for $X = 0$.

3.5 Physical picture of the basic equation

The basic equation (3.4.45) describes the dynamics of the elements of the reduced density matrix by the form of the tri-diagonal differential equation. The relaxation process of the strongly coupled system is represented by the dynamics of these elements of the reduced density matrix. We consider the physical picture of the dynamics of the elements.

Since we have used the eigenstates of the system Hamiltonian (3.2.1) as the bases, the reduce density matrix evolves in time to become a diagonalized form at $t = \infty$. That is, the off-diagonal elements decay in time and the diagonal elements relax to construct the canonical distribution of the relevant coupled system given by

$$\begin{aligned} \rho_{eq} &= \frac{1}{Z}\sum_n \left(\mathrm{e}^{-E_+^n/k_BT} \left|\varphi\left(n,1\right)\right\rangle \left\langle\varphi\left(n,1\right)\right| + \mathrm{e}^{-E_-^n/k_BT} \left|\varphi\left(n,2\right)\right\rangle \left\langle\varphi\left(n,2\right)\right| \right) \\ &+ \frac{\mathrm{e}^{\hbar\omega_0/2k_BT}}{Z} + \left|\varphi\left(0,-1\right)\right\rangle \left\langle\varphi\left(0,-1\right)\right| \end{aligned} \tag{3.5.1}$$

where Z is the partition function given by

$$Z = \sum_n \left(\mathrm{e}^{-E_+^n/k_BT} + \mathrm{e}^{-E_-^n/k_BT} \right) + \mathrm{e}^{\hbar\omega_0/2k_BT} \tag{3.5.2}$$

at the thermal equilibrium of temperature T. Thus, the decay of the off-diagonal elements (decoherence process) represents the phase relaxation and transitions among the diagonal elements to construct the canonical distribution (diagonal process) representing the energy relaxation. Quantum mechanical motion arising from the first term of the right hand side of (3.3.6) is non-vanishing for the off-diagonal part. Therefore, the decoherence process is intimately related with disappearance of the quantum mechanical motion.

It is the very characteristic point of (3.4.45) that the time evolution of the vector $\boldsymbol{\rho}^{nm}$ is entirely determined by $\boldsymbol{\rho}^{n-1\ m-1}$, $\boldsymbol{\rho}^{nm}$ and $\boldsymbol{\rho}^{n+1\ m+1}$, namely, the *transition* dynamics of $\boldsymbol{\rho}^{nm}$ is confined within the subspaces specified by $(n-1, m-1)$, (n, m) and $(n+1, m+1)$.

We should note that the elements of the reduced density matrix are characterized by two kinds of quantum numbers, the boson quantum numbers n, m for $n, m \geq -1$

($n = -1$ and $m = -1$ are defined by (3.4.6)-(3.4.8)) and the spin quantum numbers α, β ($\alpha, \beta = 1, 2$). The diagonal elements $\rho_{\alpha\alpha}^{nn}$ are diagonal in the both quantum numbers. We see from (3.4.45) that the off-diagonal part and the diagonal part in the boson quantum number is not related. But the decoherence process and the diagonal process are dependent each other since the vector $\boldsymbol{\rho}^{nm}$ contains the off-diagonal elements in the spin quantum number. In other words, the phase relaxation and the energy relaxation is correlated only through the elements of the diagonal part in the boson quantum number.

In some cases, we find that the system does not relax to construct the canonical distribution. If the system interacts with the reservoir only by the form (3.3.4), the *transition* to the different subspaces are forbidden and the system can not relax to form the canonical distribution. This kind of interaction constructs the canonical distribution only in a subspace. This suppression of *transition* also occurs when energy levels are degenerate preventing construction of the canonical distribution [9].

Expectation values are also given with the use of the elements of the reduced density matrix. The following expectation values are related with elements of the diagonal part in the boson quantum numbers:

$$\begin{aligned}\langle S_z \rangle_t &= \frac{1}{2}\sum_n \{ -\cos 2\theta_n \left(\rho_{11}^{nn}(t) - \rho_{22}^{nn}(t)\right) \\ &+ \sin 2\theta_n \left(\rho_{12}^{nn}(t) + \rho_{21}^{nn}(t)\right) - \rho_{00}(t) \}, \end{aligned} \tag{3.5.3}$$

$$\begin{aligned}\langle b^\dagger b \rangle_t &= \sum_n \left\{ \left(n + \cos^2\theta_n\right) \rho_{11}^{nn}(t) \right. \\ &- \cos\theta_n \sin\theta_n \left(\rho_{12}^{nn}(t) + \rho_{21}^{nn}(t)\right) \\ &+ \left. \left(n + \sin^2\theta_n\right) \rho_{22}^{nn}(t) \right\}, \end{aligned} \tag{3.5.4}$$

$$\begin{aligned}\langle b^\dagger b b^\dagger b \rangle_t &= \sum_n \left\{ \left(n^2 + (2n+1)\cos^2\theta_n\right) \rho_{11}^{nn}(t) \right. \\ &- (2n+1)\cos\theta_n \sin\theta_n \left(\rho_{12}^{nn}(t) + \rho_{21}^{nn}(t)\right) \\ &+ \left. \left(n^2 + (2n+1)\sin^2\theta_n\right) \rho_{22}^{nn}(t) \right\} \end{aligned} \tag{3.5.5}$$

and

$$\langle bS_+ \rangle_t = \sum_n \{ h_{11}(n)\, \rho_{11}^{nn}(t) + h_{21}(n)\, \rho_{12}^{nn}(t)$$

$$+ \quad h_{12}(n)\,\rho_{21}^{nn}(t) + h_{22}(n)\,\rho_{22}^{nn}(t)\Big\}\,. \tag{3.5.6}$$

With the elements of the off-diagonal part in the boson quantum number, the following expectation values are given:

$$\begin{aligned}
\langle S_+\rangle_t &= \sum_n \Big\{ f_{11}(n)\,\rho_{11}^{n\;n+1}(t) + f_{21}(n)\,\rho_{12}^{n\;n+1}(t) \\
&+ f_{12}(n)\,\rho_{21}^{n\;n+1}(t) + f_{22}(n)\,\rho_{22}^{n\;n+1}(t)\Big\} \\
&+ f_{10}\,\rho_{01}^{0}(t) + f_{20}\,\rho_{02}^{0}(t)\,,
\end{aligned} \tag{3.5.7}$$

$$\begin{aligned}
\langle b^\dagger\rangle_t &= \sum_n \Big\{ d_{11}(n)\,\rho_{11}^{n\;n+1}(t) + d_{21}(n)\,\rho_{12}^{n\;n+1}(t) \\
&+ d_{12}(n)\,\rho_{21}^{n\;n+1}(t) + d_{22}(n)\,\rho_{22}^{n\;n+1}(t)\Big\} \\
&+ d_{10}\,\rho_{01}^{0}(t) + d_{20}\,\rho_{02}^{0}(t)\,.
\end{aligned} \tag{3.5.8}$$

Above expressions of the expectation values are direct reflection of the operator characteristics on the eigenstates as mentioned in the previous section.

3.6 Calculation of dynamics

3.6.1 Initial conditions

We assume that the spin subsystem, the boson subsystem and the reservoir are prepared independently at $t = 0$. Especially, the reservoir is considered to be in an equilibrium with temperature T (the canonical distribution) and the spin subsystem is in the up state. For the boson subsystem, we impose the following initial conditions:

Fock state

The boson subsystem is in a Fock state, that is, n boson state. The reduced density matrix at the initial time $t = 0$ is represented by

$$\begin{aligned} \rho(0) &= \sin^2\theta_n \left|\varphi(n,1)\right\rangle\left\langle\varphi(n,1)\right| + \cos^2\theta_n \left|\varphi(n,2)\right\rangle\left\langle\varphi(n,2)\right| \\ &+ \sin\theta_n\cos\theta_n \left(\left|\varphi(n,1)\right\rangle\left\langle\varphi(n,2)\right| + \left|\varphi(n,2)\right\rangle\left\langle\varphi(n,1)\right|\right). \end{aligned} \tag{3.6.1}$$

Thermal state

The boson subsystem is in the thermal equilibrium state of temperature T_0 with canonical distribution. The initial reduced density matrix $\rho(0)$ is written in the form

$$\begin{aligned} \rho(0) &= \frac{1}{Z_b}\sum_n \mathrm{e}^{-\hbar\omega_b n/k_B T_0}\Big(\sin^2\theta_n \left|\varphi(n,1)\right\rangle\left\langle\varphi(n,1)\right| \\ &+ \sin\theta_n\cos\theta_n \left(\left|\varphi(n,1)\right\rangle\left\langle\varphi(n,2)\right| + \left|\varphi(n,2)\right\rangle\left\langle\varphi(n,1)\right|\right) \\ &+ \cos^2\theta_n \left|\varphi(n,2)\right\rangle\left\langle\varphi(n,2)\right|\Big) \end{aligned} \tag{3.6.2}$$

where

$$Z_b = \sum_{n=0} \mathrm{e}^{-\hbar\omega_b n/k_B T_0}. \tag{3.6.3}$$

Coherent state

The boson subsystem is in a coherent state $|z\rangle$ defined by $b\ |z\rangle = z\ |z\rangle$. Then the initial reduced density matrix $\rho(0)$ is written by

$$\rho(0) = |z\rangle\langle z|$$

$$
\begin{aligned}
&= \sum_{n,m} \mathrm{e}^{-|z|^2} \frac{z^n z^{*m}}{\sqrt{n!m!}} \left(\sin\theta_n \sin\theta_m \left|\varphi\left(n,1\right)\right\rangle \left\langle\varphi\left(m,1\right)\right| \right. \\
&+ \sin\theta_n \cos\theta_m \left|\varphi\left(n,1\right)\right\rangle \left\langle\varphi\left(m,2\right)\right| \\
&+ \cos\theta_n \sin\theta_m \left|\varphi\left(n,2\right)\right\rangle \left\langle\varphi\left(m,1\right)\right| \\
&+ \left. \cos\theta_n \cos\theta_m \left|\varphi\left(n,2\right)\right\rangle \left\langle\varphi\left(m,2\right)\right|\right) . \qquad (3.6.4)
\end{aligned}
$$

As is seen from (3.6.1) and (3.6.2), elements of the off-diagonal part in the boson quantum number do not appear in $\rho\left(0\right)$ if we impose the Fock state and the thermal state initial condition for the boson subsystem. That is, we have $\rho_{\alpha\beta}^{nm}\left(t\right) = 0$ for $n \neq m$ for these initial conditions. On the other hand, the coherent state initial condition for the boson subsystem requires calculations of the off-diagonal part as well as the diagonal part in the boson quantum numbers. However, as we have shown in the previous sections, time evolution of each part is not correlated. Thus we can calculate each part independently.

3.6.2 Numerical calculations

In this section, we show dynamical time evolution of the diagonal part in the boson quantum number from the coherent initial condition (3.6.4). We have solved the basic equation (3.4.45) numerically by the Runge Kutta method and determined time evolution of the matrix elements. To observe decay of the off-diagonal elements and transition among diagonal elements, we calculate absolute values of each element of the reduced density matrix. We display histogram of sequence of the absolute values of the matrix elements at time τ:

$$
\left\{\rho_{00}, \rho_{11}^{00}, \rho_{12}^{00}, \rho_{21}^{00}, \rho_{22}^{00}, \rho_{11}^{11}, \rho_{12}^{11}, \rho_{21}^{11}, \rho_{22}^{11}, \ \ldots\right\}_{\tau} . \qquad (3.6.5)
$$

The $(4n+3)$th and $(4n+4)$th components for $n \geq 0$ of the sequence represent the off-diagonal elements and other components represent the diagonal elements. We also show the quantities $\langle S_z\rangle_\tau$ and $\langle b^\dagger b\rangle_\tau$ as functions of the scaled time variable $\tau = \kappa t$.

In Fig.3.1-a and in Fig.3.1-b, we show time evolution of the expectation values $\langle S_z\rangle_\tau$ and $\langle b^\dagger b\rangle_\tau$ in the short time regime and the long time regime, respectively, from the

initial condition of $z = 2.0$ in (3.6.4). The temperature parameters $\tilde{T} = k_B T/\hbar\kappa$ are set equal to 1.0 and 100.0. We show histograms of the sequence of the absolute values for several τ in Fig.3.1-c for temperature parameter $\tilde{T} = 1.0$ and in Fig.3.1-d for temperature parameter $\tilde{T} = 200.0$. The calculations are done for the damping constants, $\kappa_S = 0$, $\kappa_b = 1.0$ and $\kappa_i = 0$, that is, the dissipation from the the boson subsystem is taken into account and the other dissipation mechanisms are totally ignored. For the system parameters $\tilde{g}_\perp = g_\perp/\kappa = 100.0$, $\tilde{g}_{||} = g_{||}/\kappa = 0$, $\tilde{\omega}_0 = \omega_0/\kappa = 200.0$, $\tilde{\omega}_b = \omega_b/\kappa = 200.0$, the energy level inversion discussed in section 2 does not occur in the ground state.

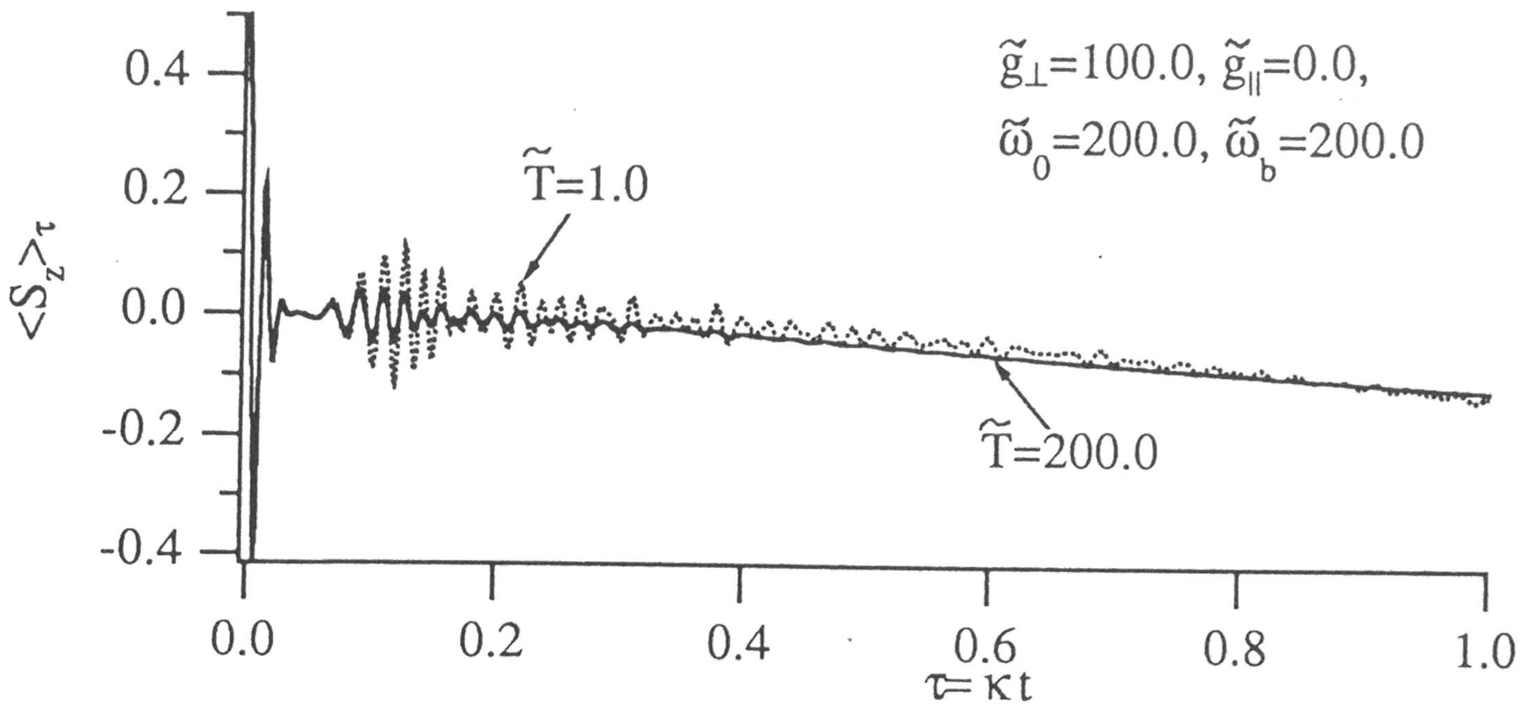
Oscillation appearing in the short time regime (Fig.3.1-a) reflects the quantum mechanical motion and superposition property. These are the typical characteristics of the Jaynes-Cummings model [2]. Earlier suppression of oscillation occurs for $\tilde{T} = 200.0$ compared with the one for $\tilde{T} = 1.0$. We see slow monotonous relaxation of $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ to the correct equilibrium values which depend on temperature in the long time regime ($\tau > 1$). In Fig.3.1-c and Fig.3.1-d we clearly see the decoherence process (decay of $(4n+3)$th and $(4n+4)$th components) in the short time regime ($0 < \tau < 1$) and transitions among the diagonal elements in the longer time scale approaching finally to the canonical distribution in the thermal equilibrium.

In Fig.3.2-a and Fig.3.2-b, we show time evolution of the expectation values from the initial condition with $z = 3.0$ in (3.6.4) for $\tilde{T} = 1.0$ and $\tilde{T} = 200.0$. The corresponding histograms are found in Fig.3.2-c for and $\tilde{T} = 1.0$ and in Fig.3.2-d for $\tilde{T} = 200.0$, respectively. The system parameters and the damping constants are the same as in Fig.3.1. Decay of matrix elements of large boson quantum number is faster than those of small quantum number. As a result, we see that the decoherence precess in Fig.3.2 takes place more rapidly compared with that in Fig.3.1.

In Fig.3.3, we show calculations for damping constants, $\kappa_S = 1.0$, $\kappa_b = 0$ and $\kappa_i = 0$, namely, the dissipation from the the spin subsystem is taken into account and the other dissipation mechanisms are totally ignored. The system parameters and the initial condition are the same as in Fig.3.1 and $\tilde{T} = 1.0$. In Fig.3.3-a and in Fig.3.3-b time evolution of the expectation values are shown in the short time regime and the

long time regime, respectively. The histograms of absolute values are shown in Fig.3.3-c. It is the characteristic point that both the phase relaxation time and the energy relaxation time are longer compared with the previous case.

In Fig.3.4, calculations with parameters $\tilde{g}_\perp = g_\perp/\kappa = 100.0$, $\tilde{g}_{||} = g_{||}/\kappa = 0$, $\tilde{\omega}_0 = \omega_0/\kappa = 50.0$, $\tilde{\omega}_b = \omega_b/\kappa = 50.0$ from the initial condition of $z = 2.0$ in (3.6.4) are shown. With these values of parameters, energy level inversion occurs in the ground state. Damping constants are $\kappa_S = 0.0$, $\kappa_b = 1.0$ and $\kappa_i = 0$ and the temperature parameter takes values $\tilde{T} = 1.0$ and $\tilde{T} = 50.0$. In the short time regime at low temperature $\tilde{T} = 1.0$, time evolution is similar to the one for Fig.3.1. On the other hand, time evolution in the long time regime is peculiar to the system parameters reflecting the transition property among diagonal elements toward thermal equilibrium determined by the energy level configuration.



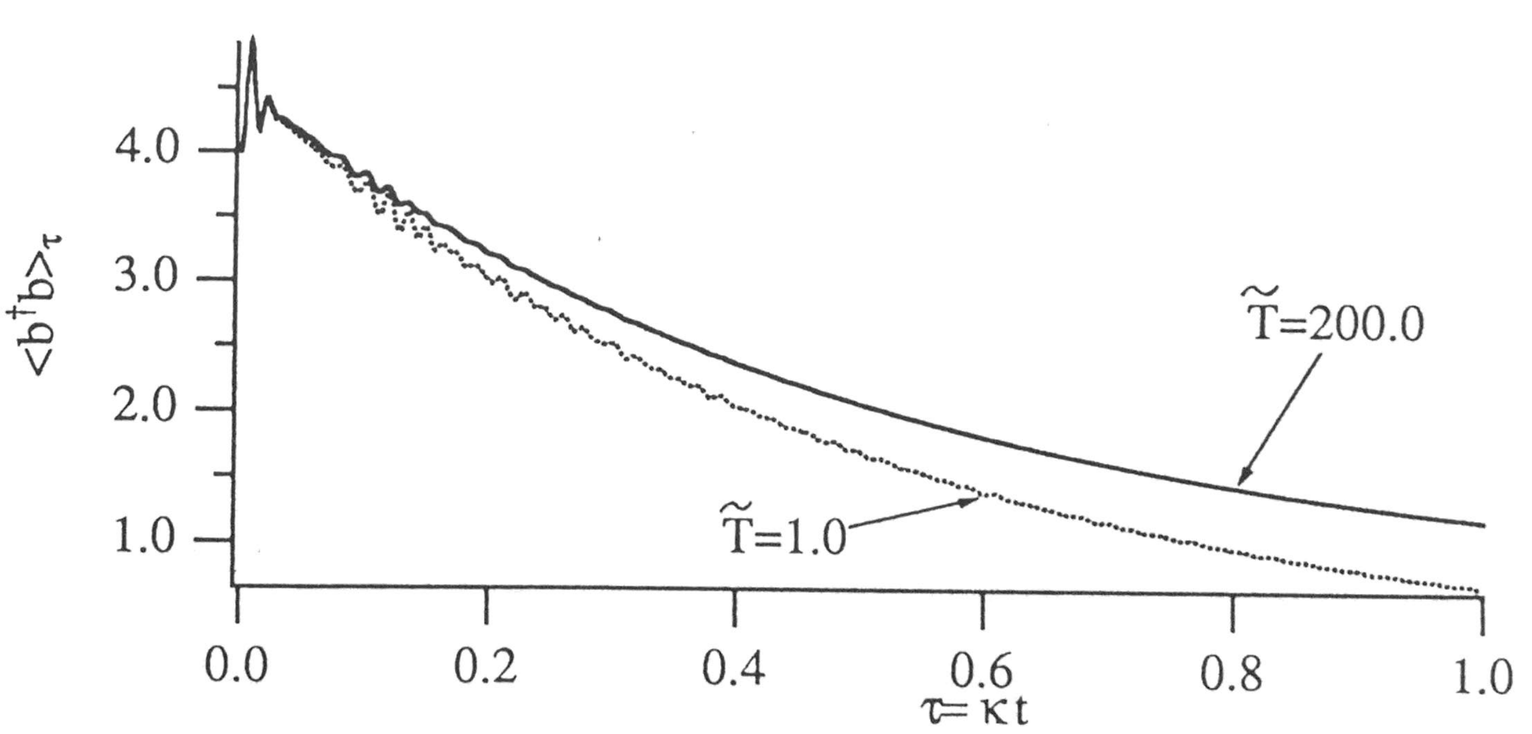
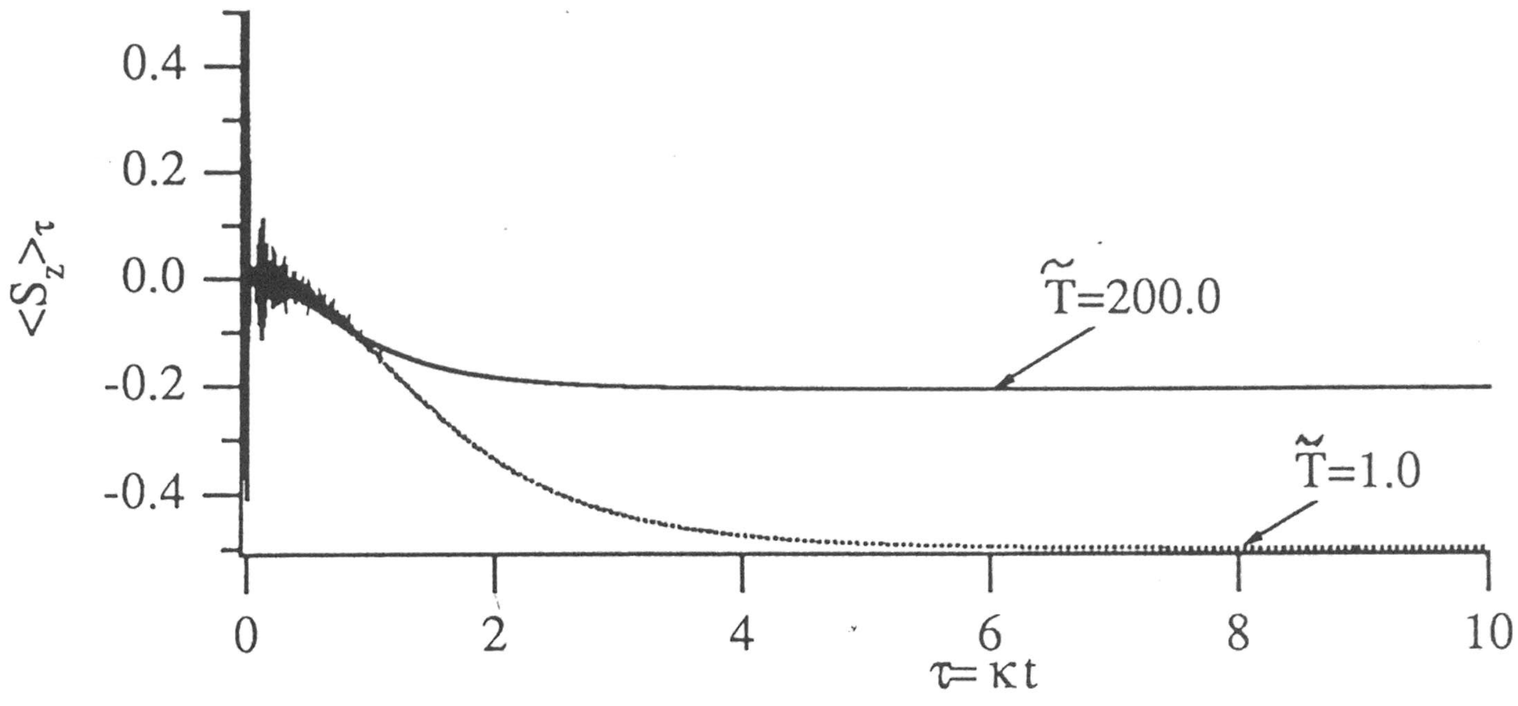


Fig.3.1-a Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the short time regime for the initial condition of $z = 2.0$. System parameters are $\tilde{g}_\perp = g_\perp/\kappa = 100.0$, $\tilde{g}_{||} = g_{||}/\kappa = 0$, $\tilde{\omega}_0 = \omega_0/\kappa = 200.0$, $\tilde{\omega}_b = \omega_b/\kappa = 200.0$ and the damping constants are $\kappa_S = 0$, $\kappa_b = 1.0$ and $\kappa_i = 0$. The temperature parameter is given by $\tilde{T} = k_B T/\hbar\kappa = 1.0$ (dashed line) and $\tilde{T} = 200.0$ (solid line).



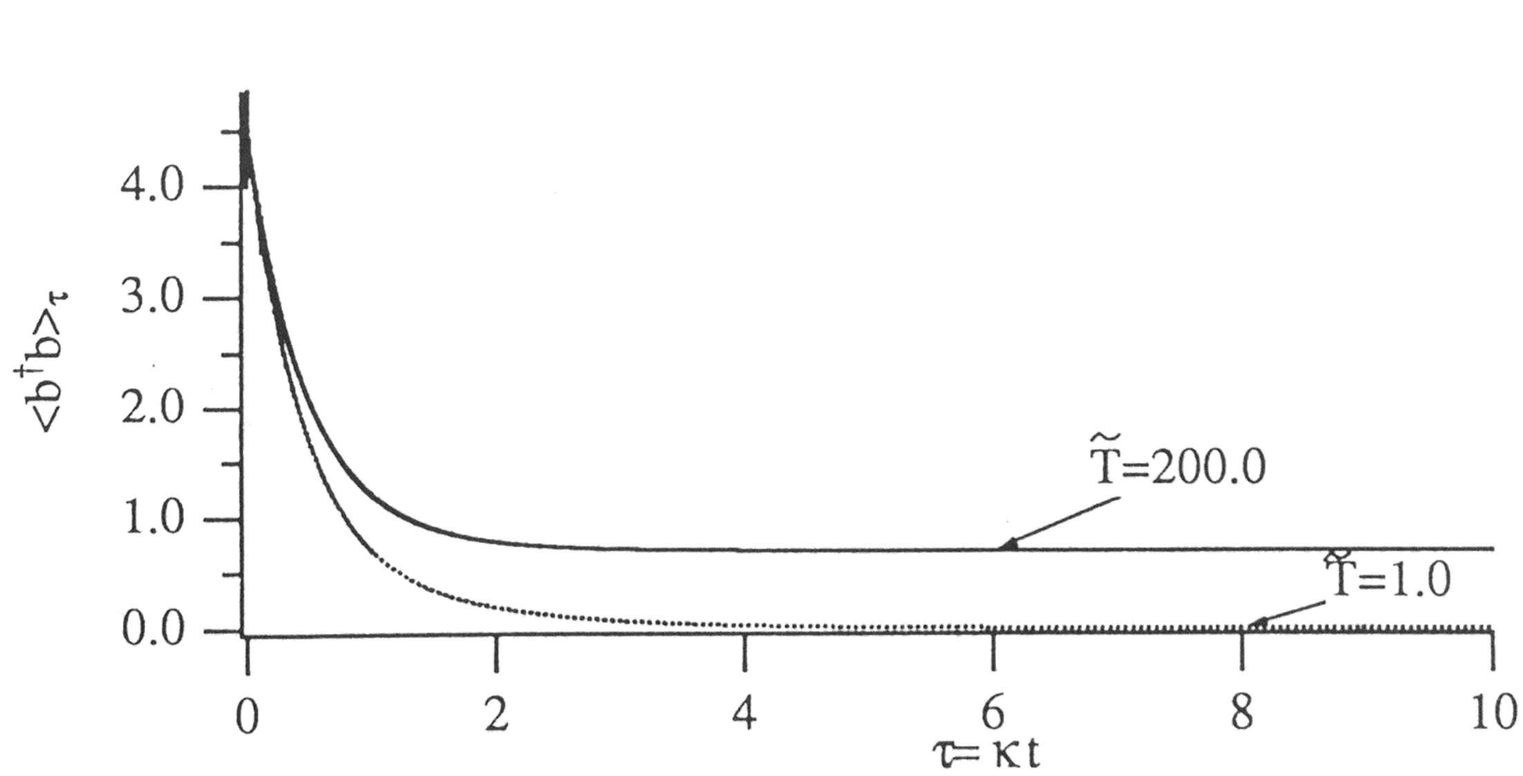
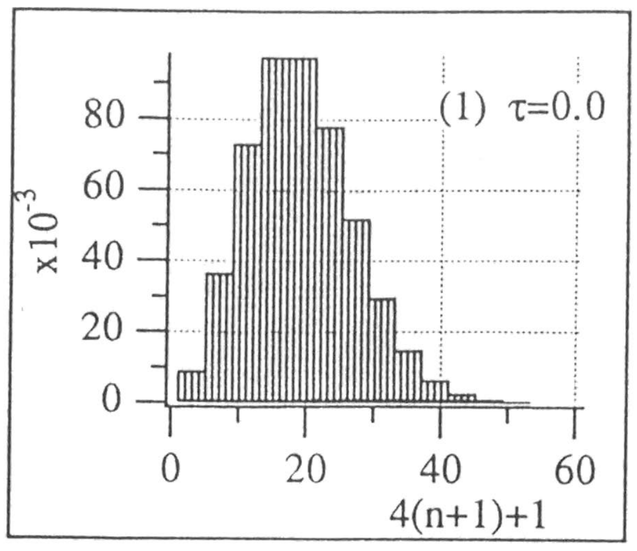
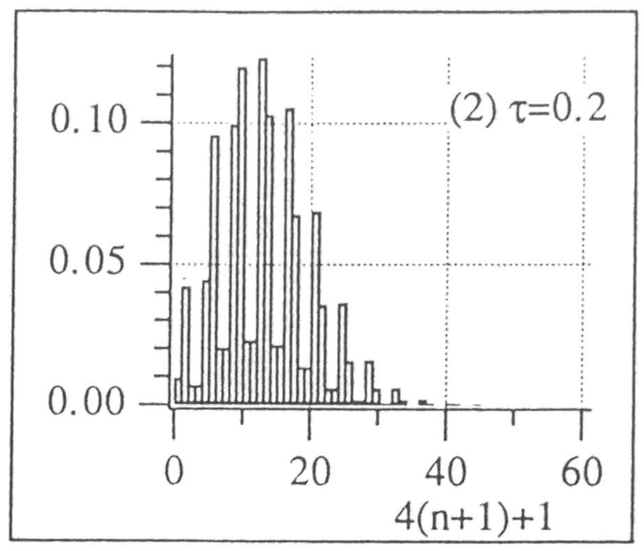
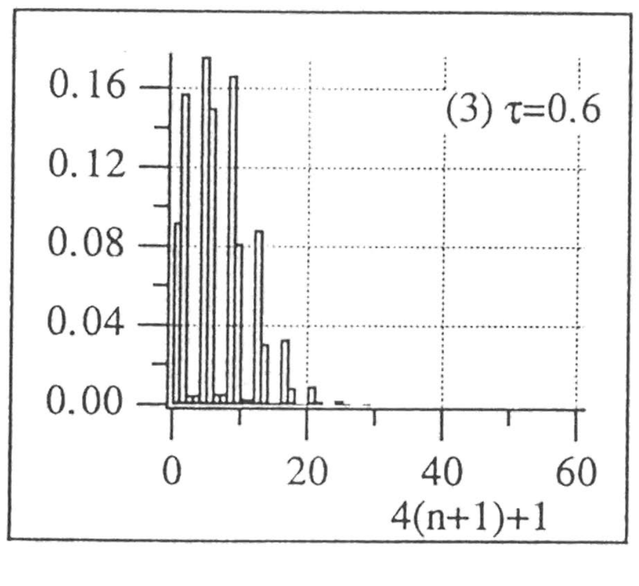
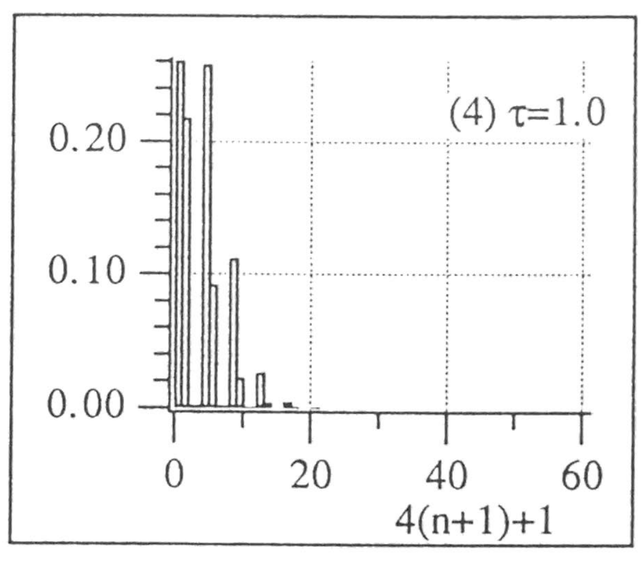


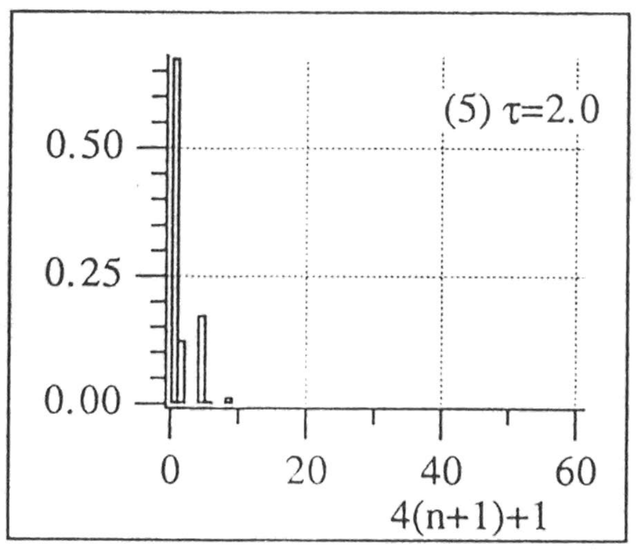
Fig.3.1-b Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the long time regime.











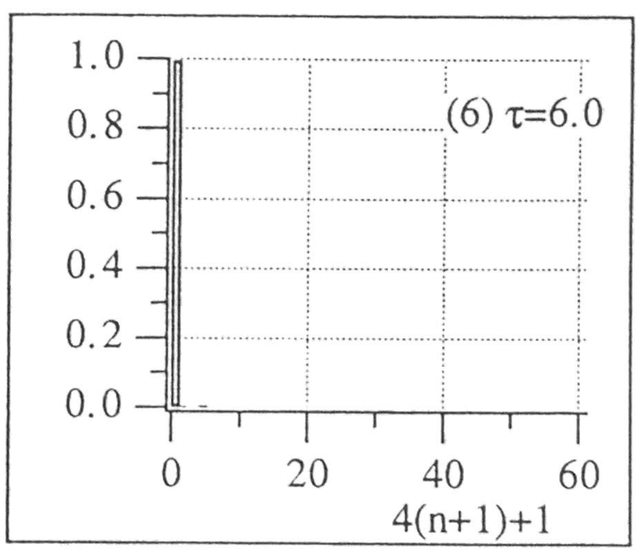


Fig.3.1-c Histograms representation of the reduced density matrix for several values of $\tau = \kappa t$; $\tilde{T} = 1.0$. See the main text for more details.

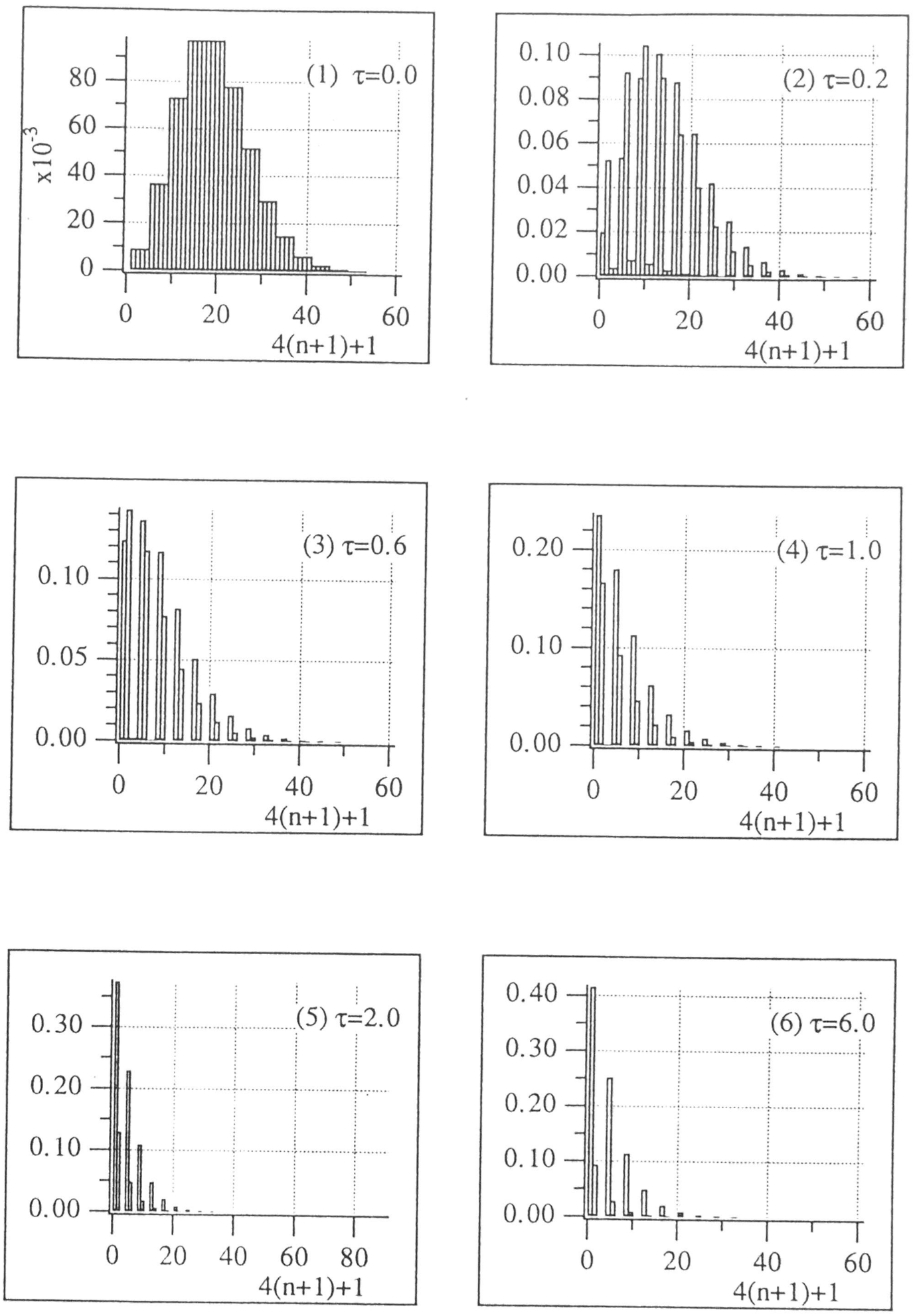
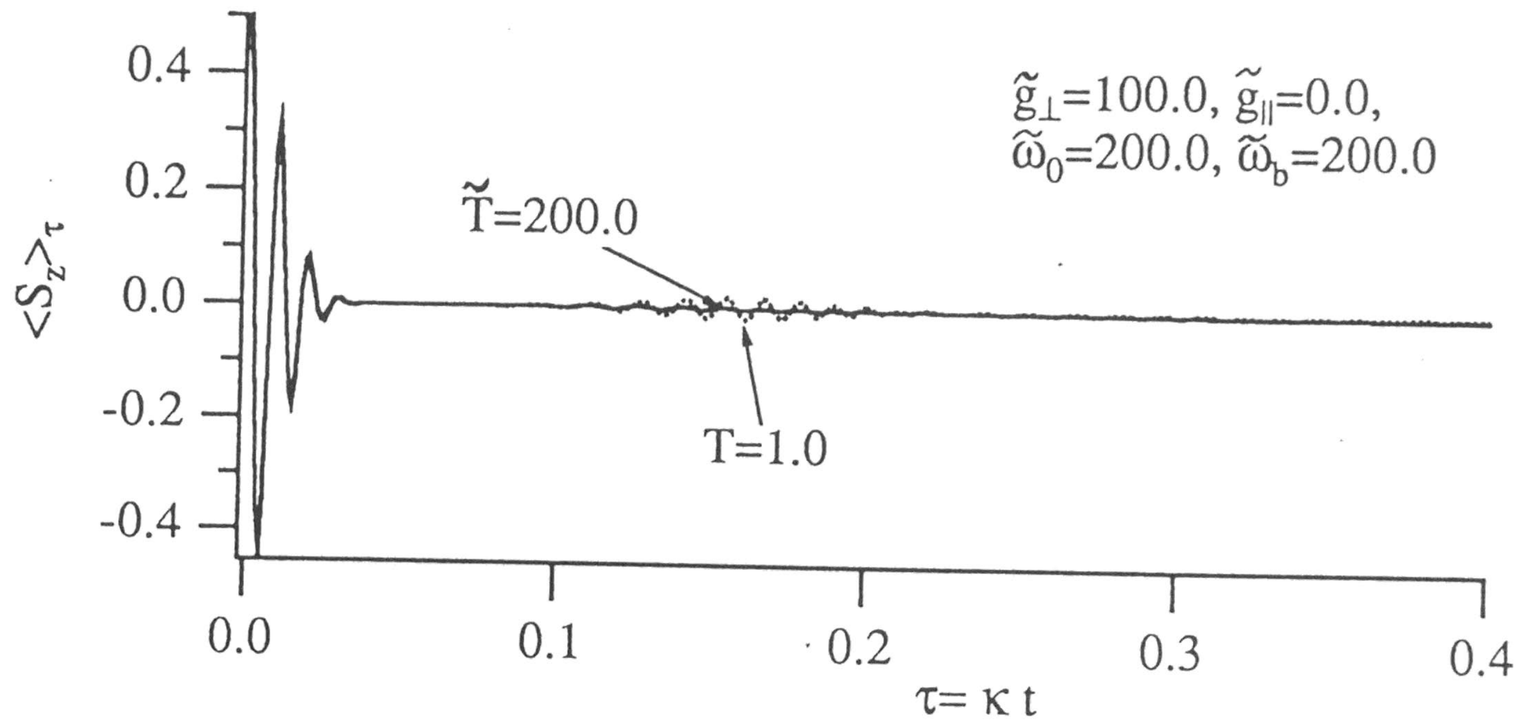


Fig.3.1-d Histograms representation of the reduced density matrix for several values of $\tau = \kappa t$; $T = 200.0$.



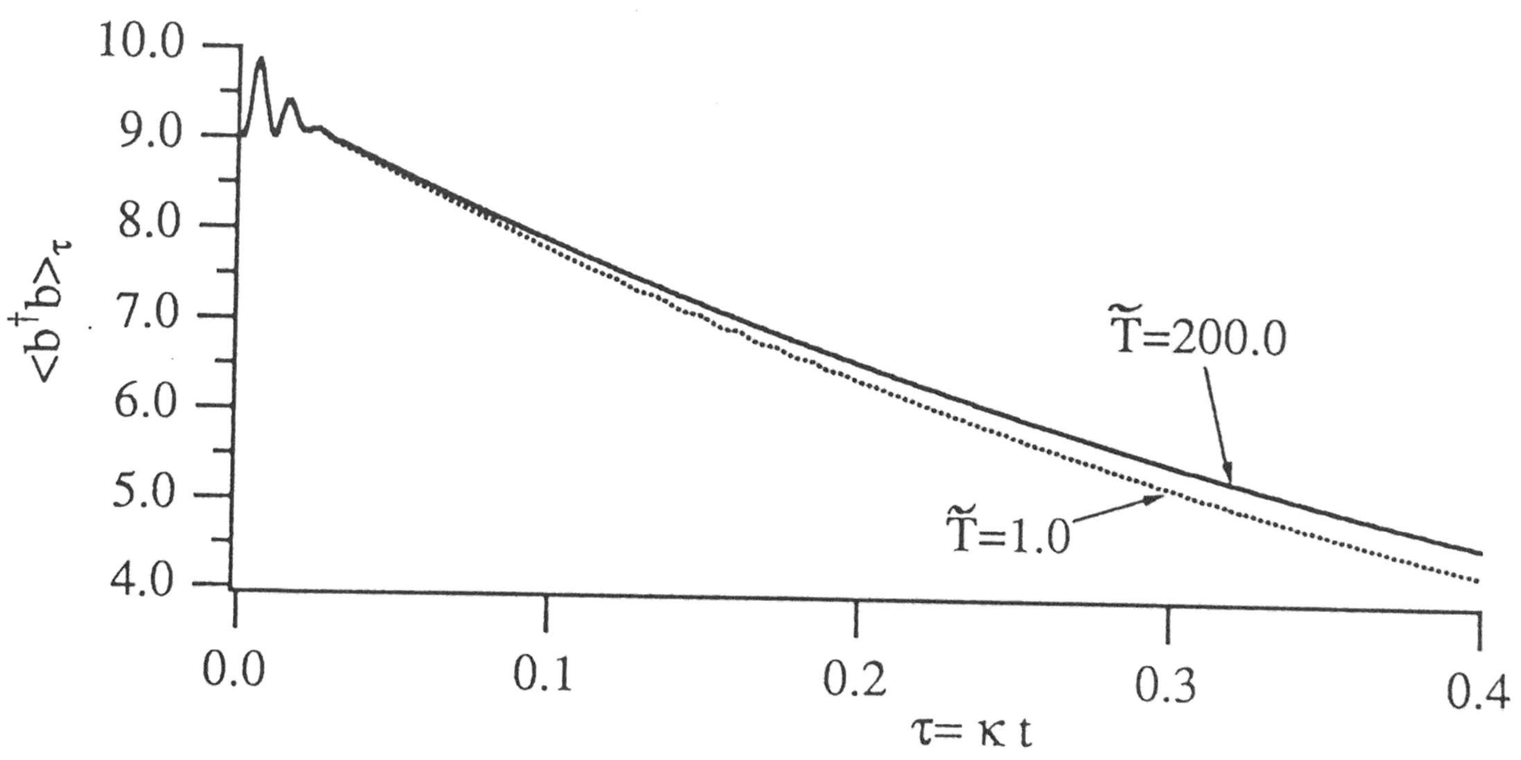
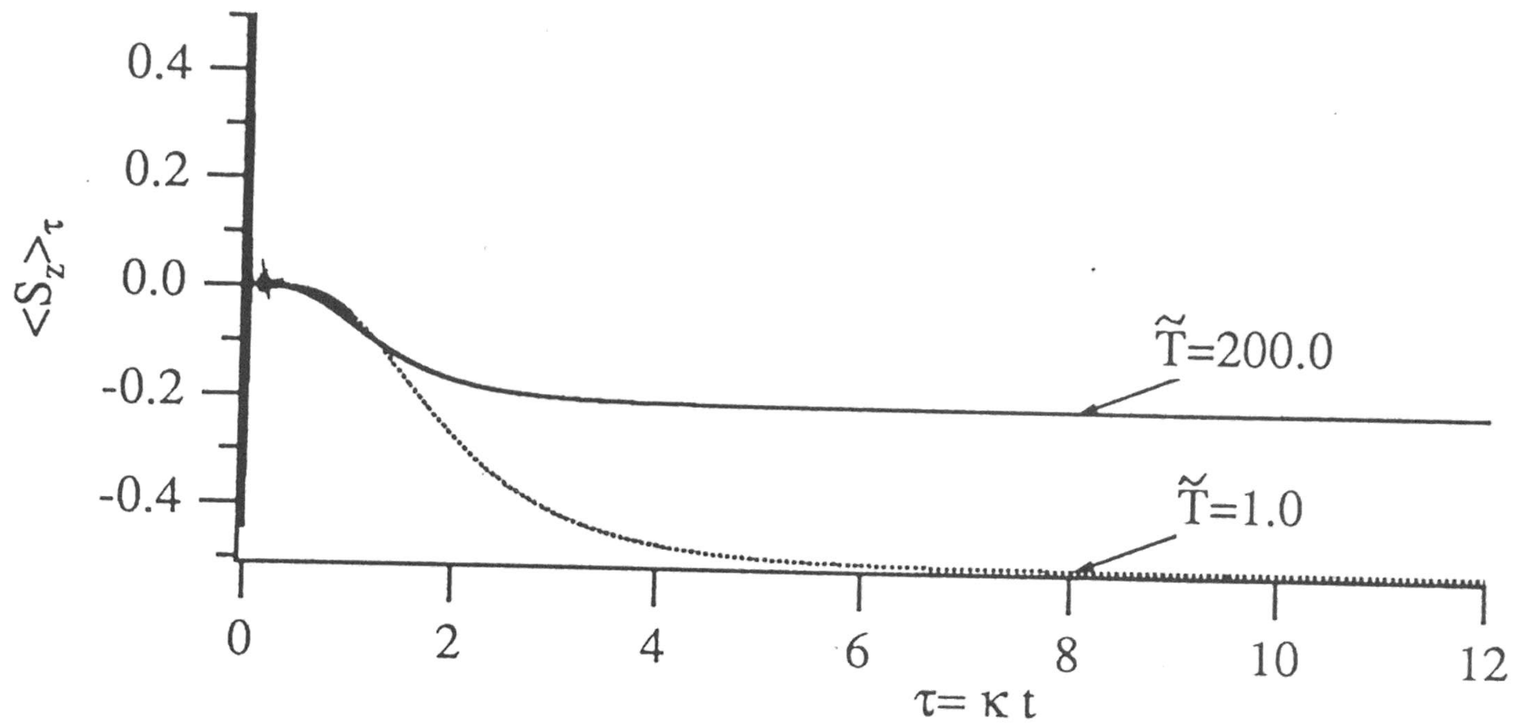


Fig.3.2-a Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the short time regime for the initial condition of $z = 3.0$. The system parameters and the damping constants are the same as in Fig.3.1; $\tilde{T} = 1.0$ (dashed line) and $\tilde{T} = 200.0$ (solid line).



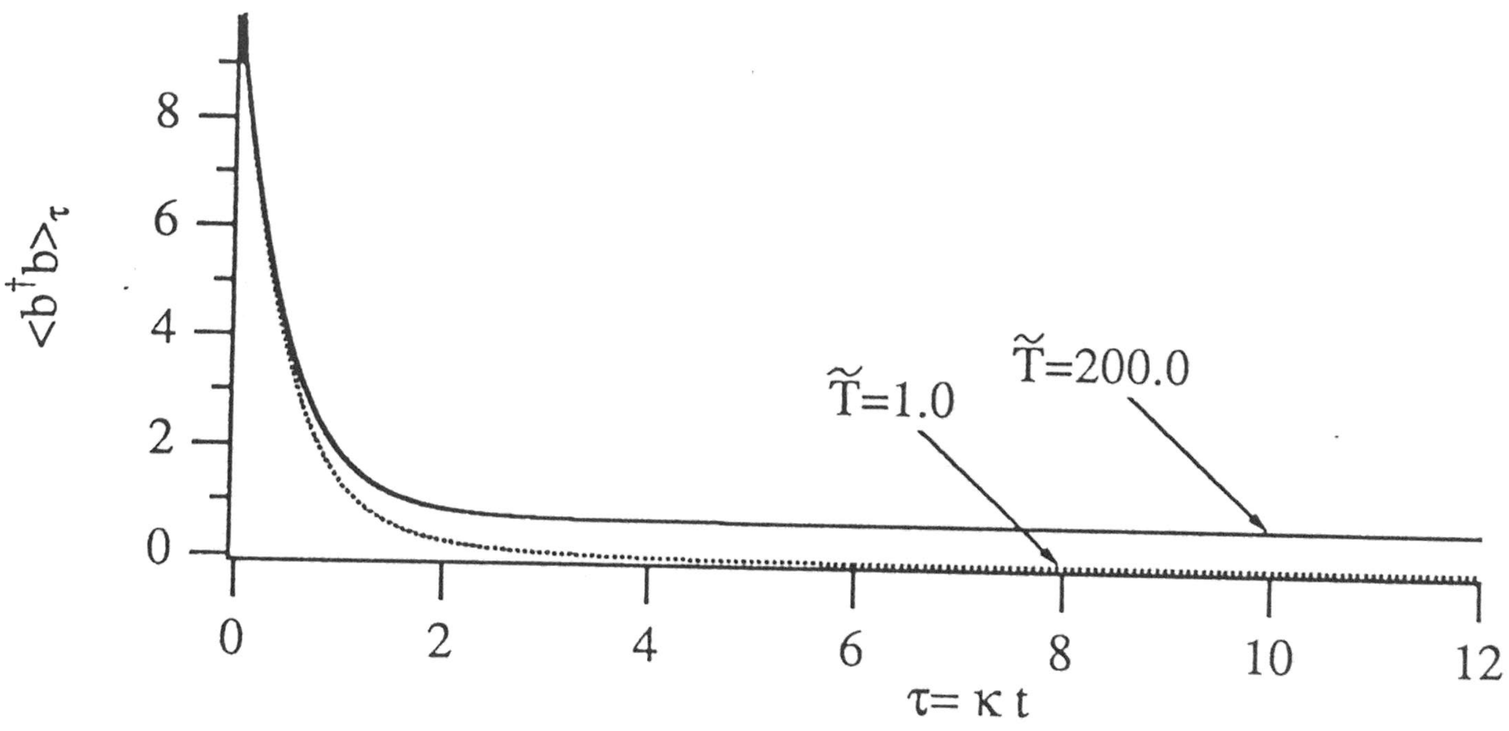
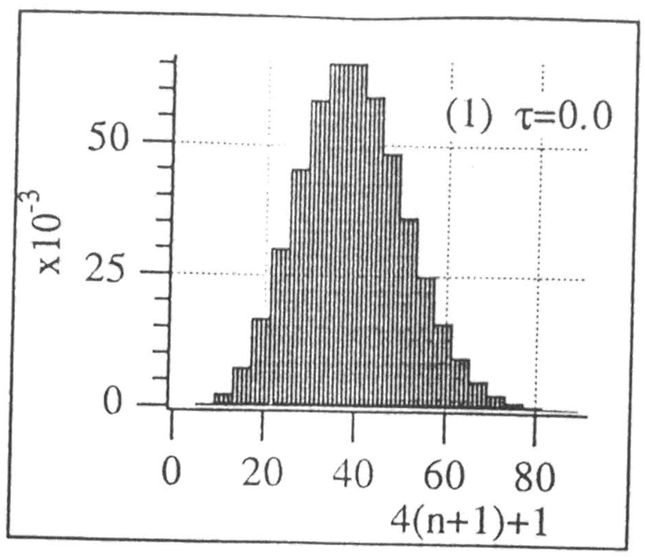
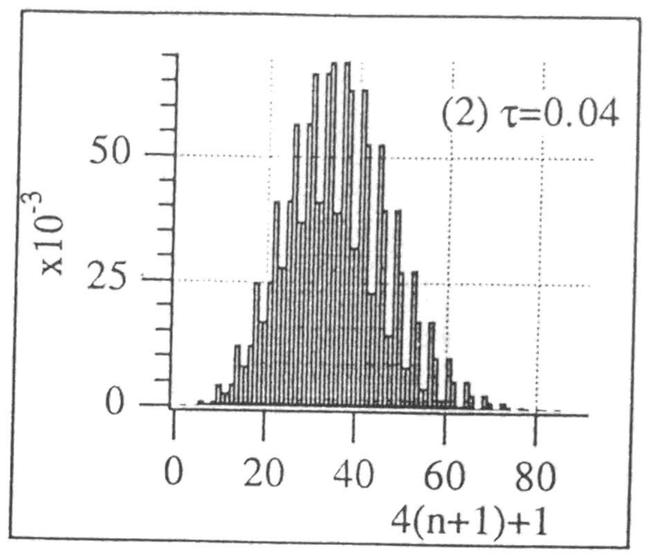
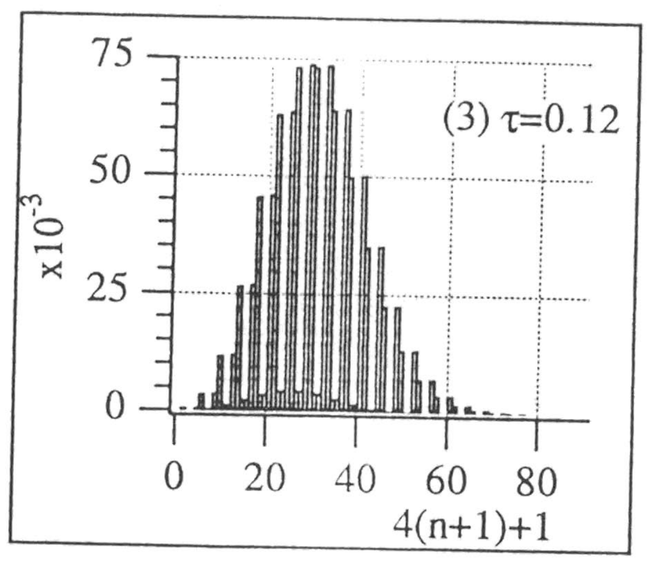
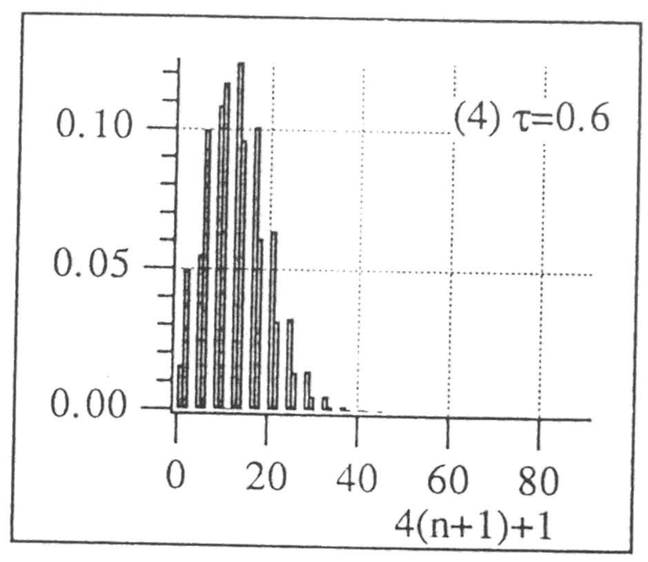


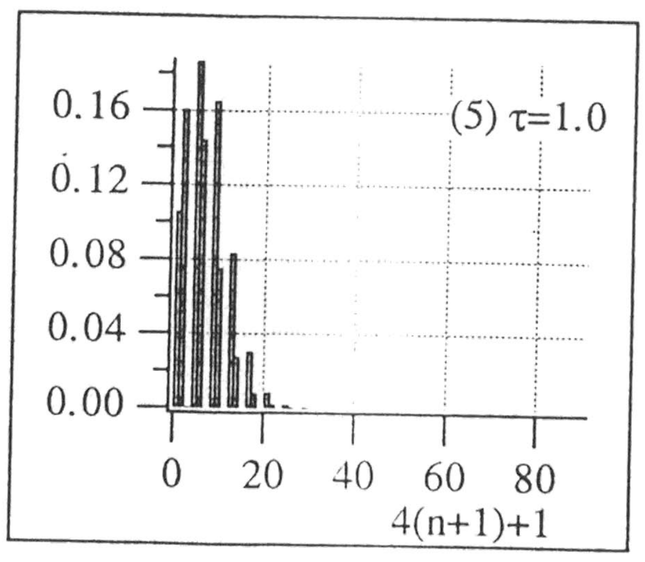
Fig.3.2-b Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the long time regime.











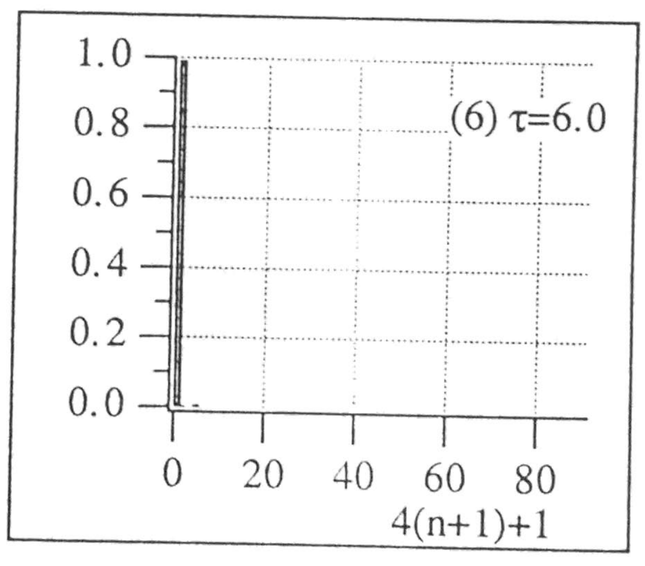
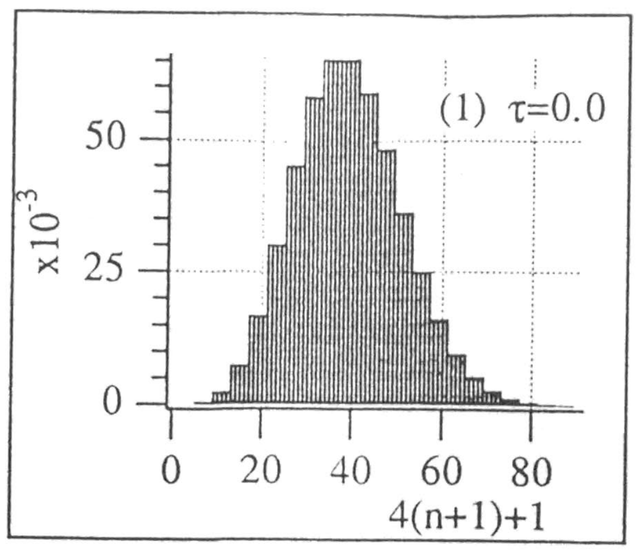
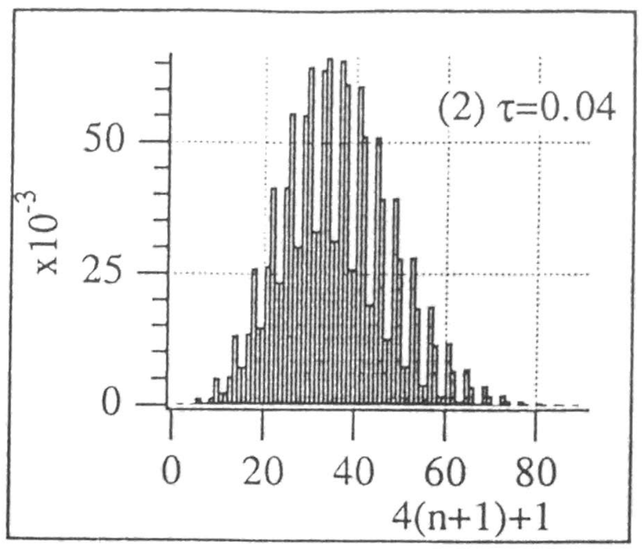
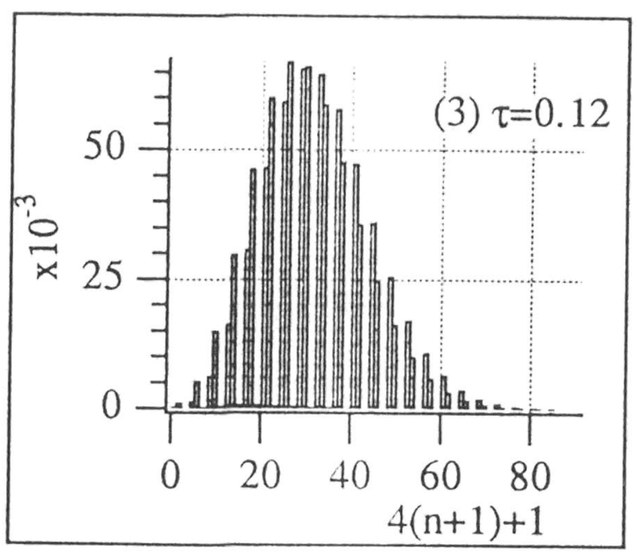
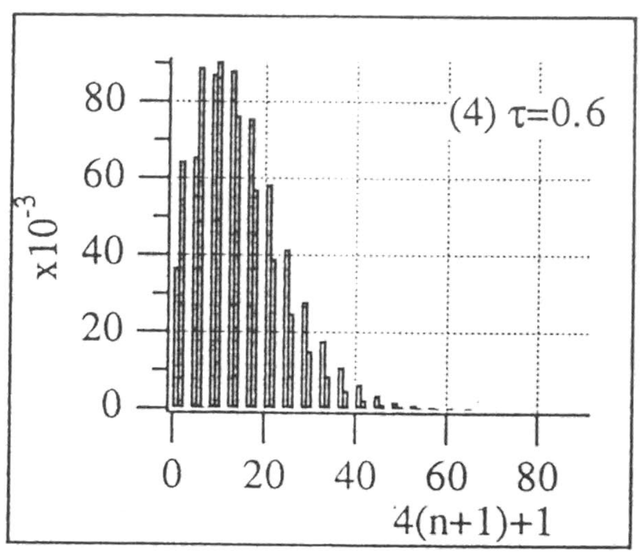


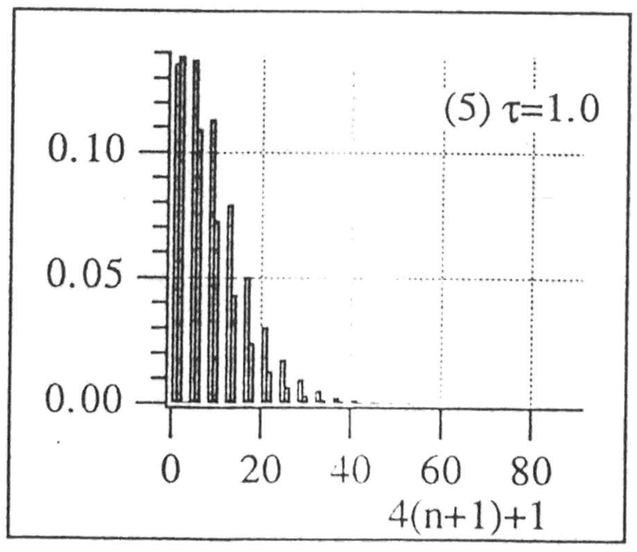
Fig.3.2-c Histograms representation of the reduced density matrix for several values of $\tau = \kappa t$; $T = 1.0$.











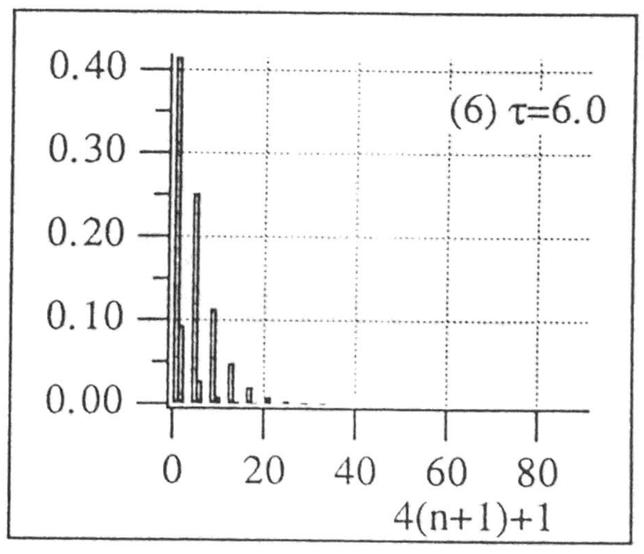
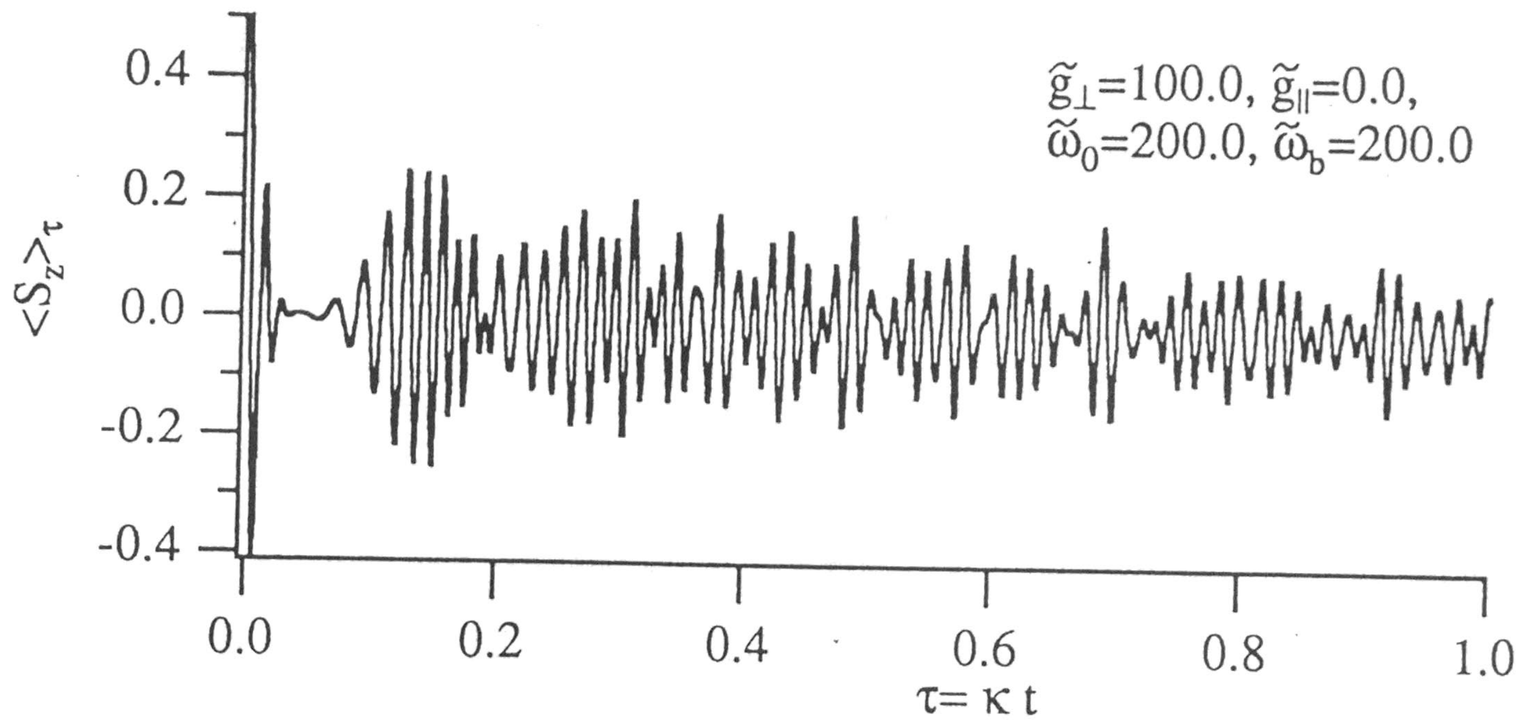


Fig.3.2-d Histograms representation of the reduced density matrix for several values of $\tau = \kappa t$; $\tilde{T} = 200.0$.



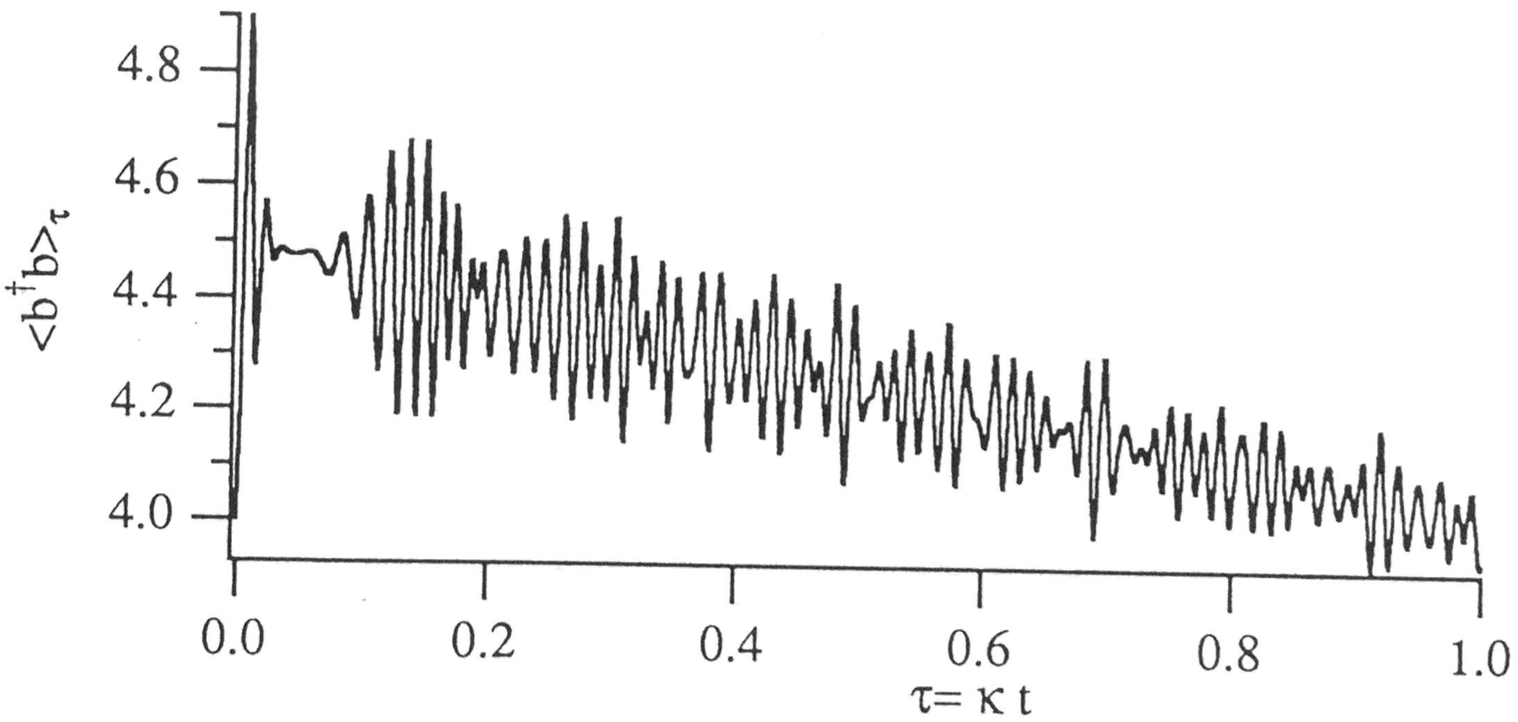
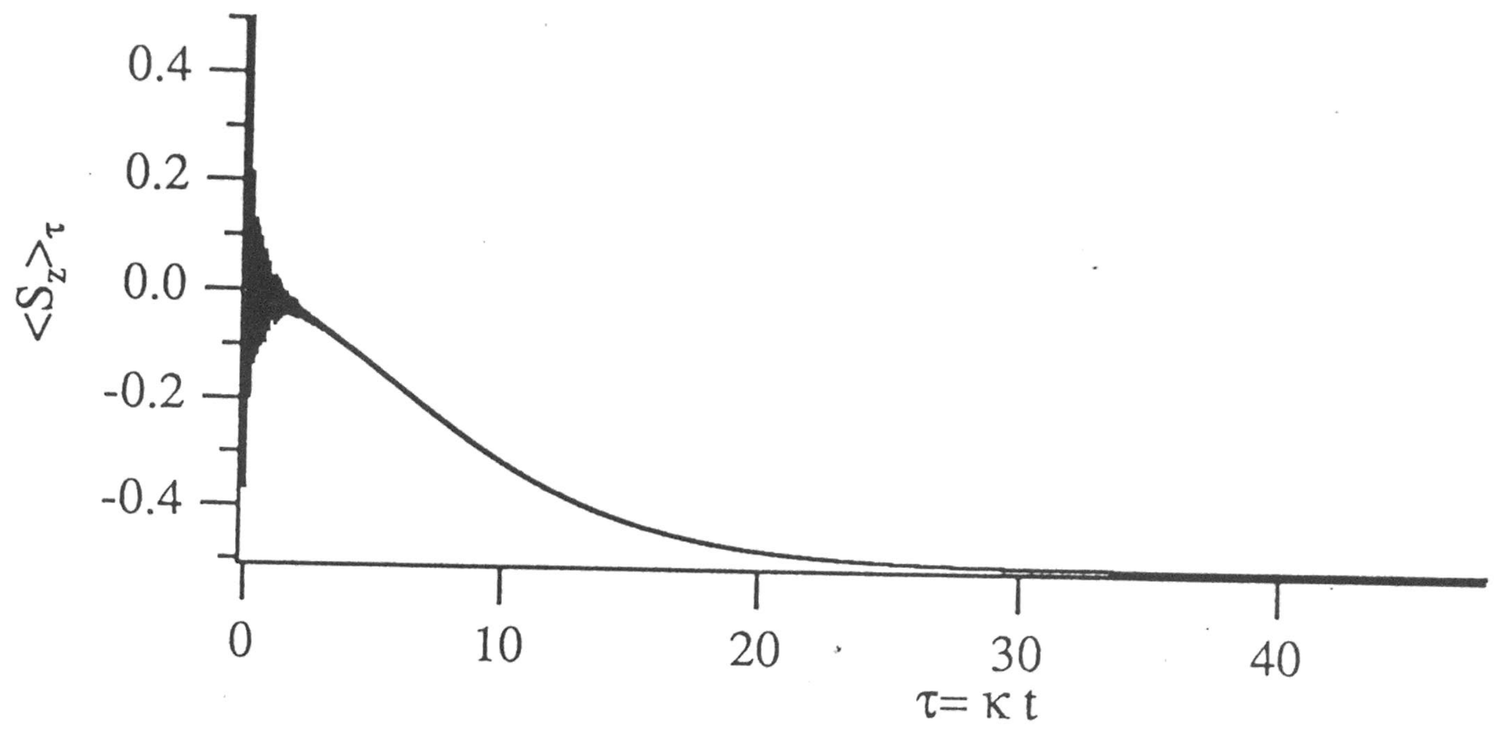


Fig.3.3-a Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the short time regime from the initial condition of $z = 2.0$ for $\tilde{T} = 1.0$. The system parameters are the same as in Fig.3.1. The damping constants are $\kappa_S = 1.0$, $\kappa_b = 0$ and $\kappa_i = 0$.



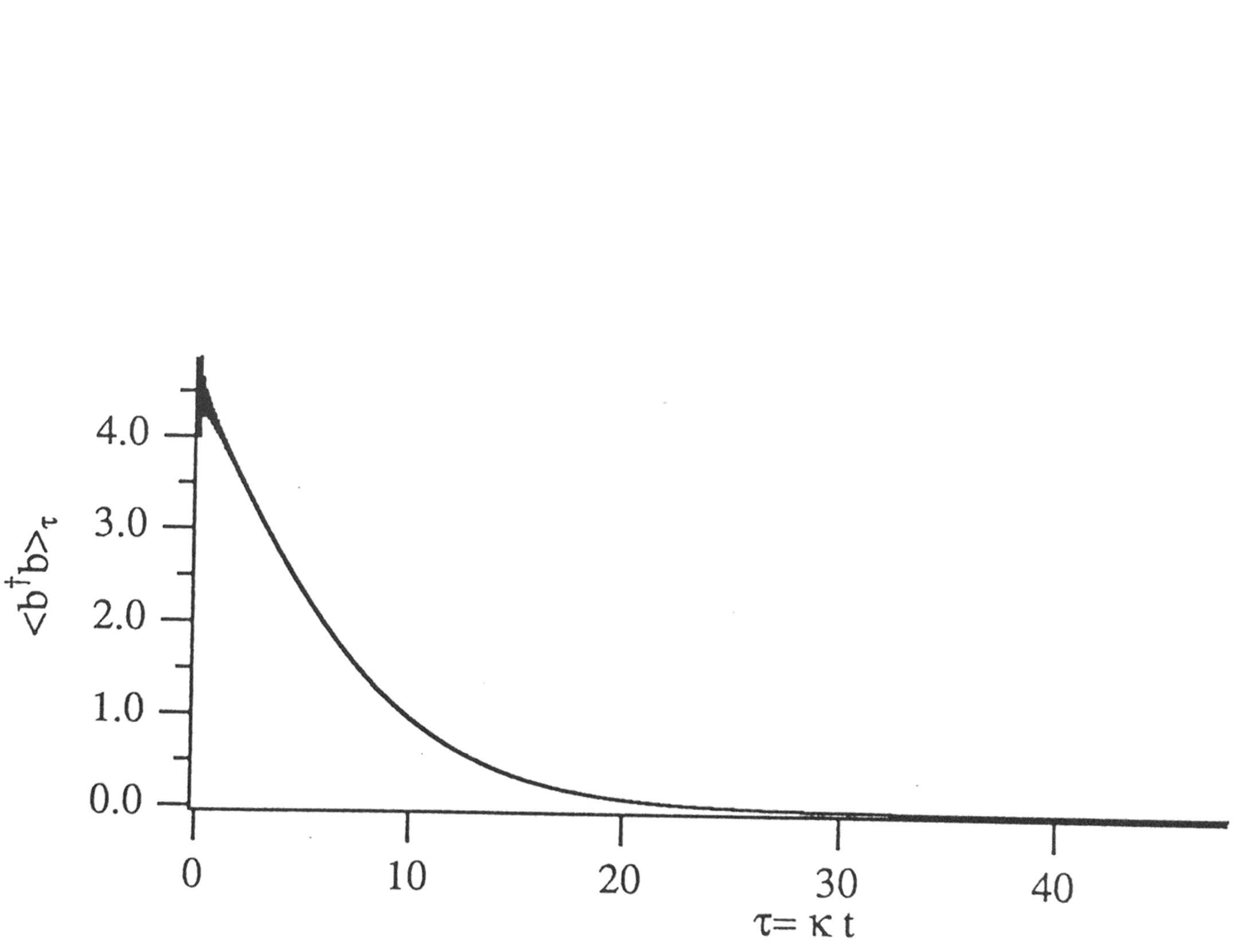
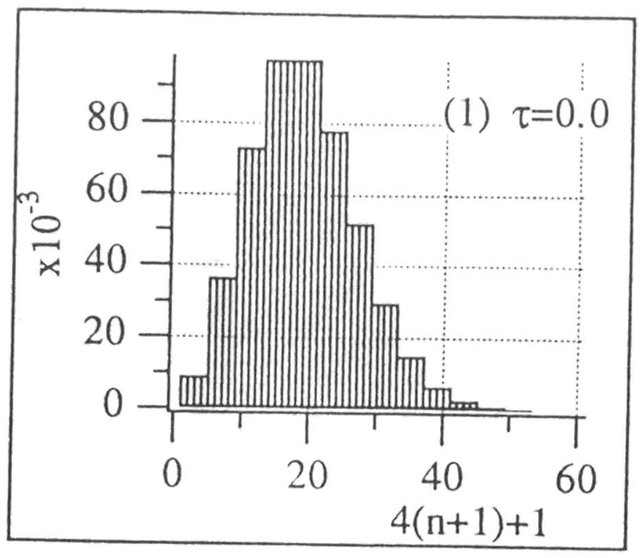
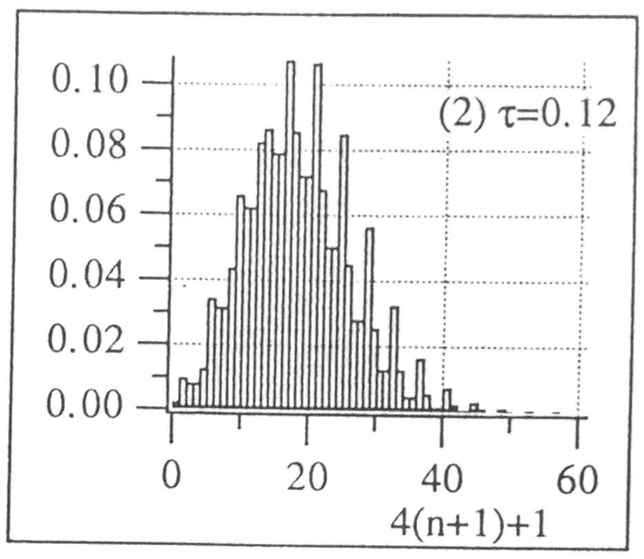
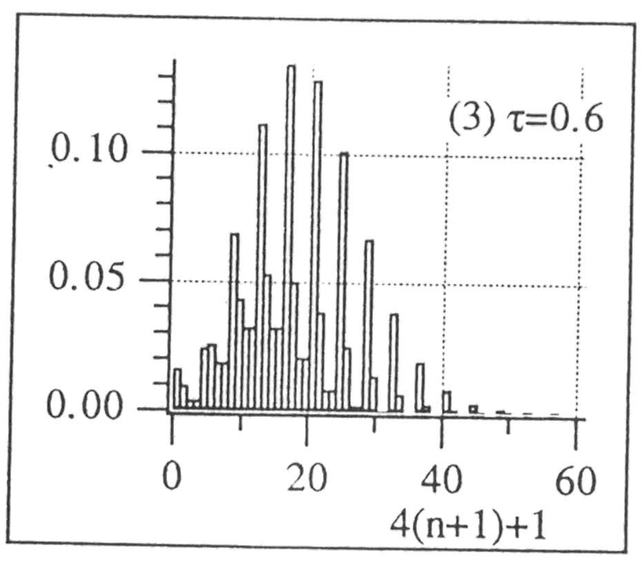
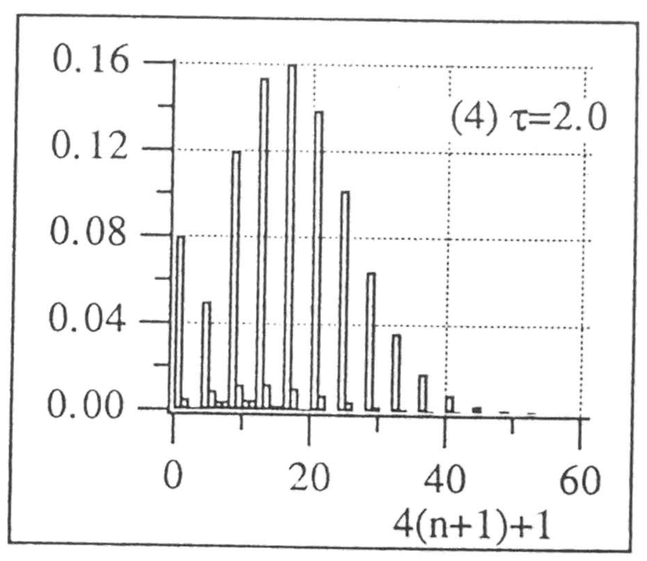


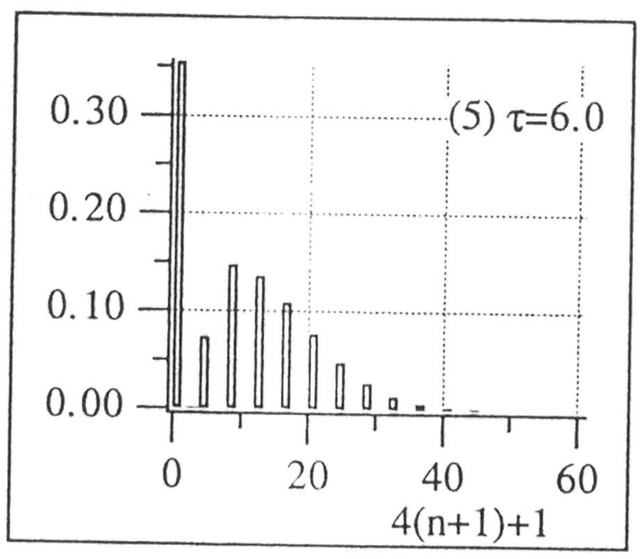
Fig.3.3-b Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the long time regime.











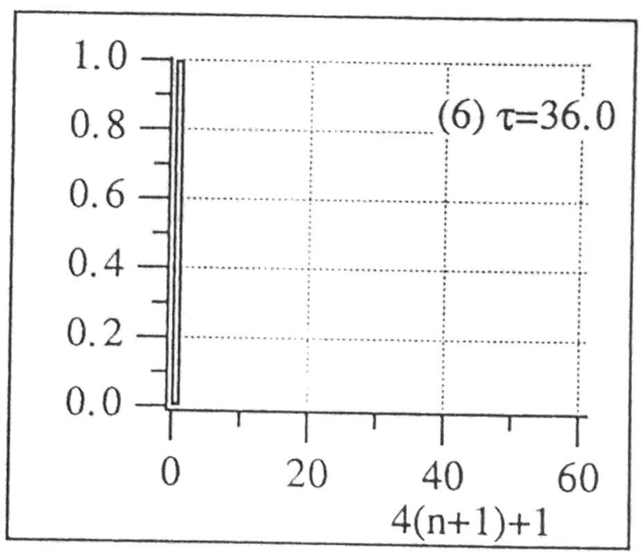


Fig.3.3-c Histograms representation of the reduced density matrix for several values of $\tau = \kappa t$.

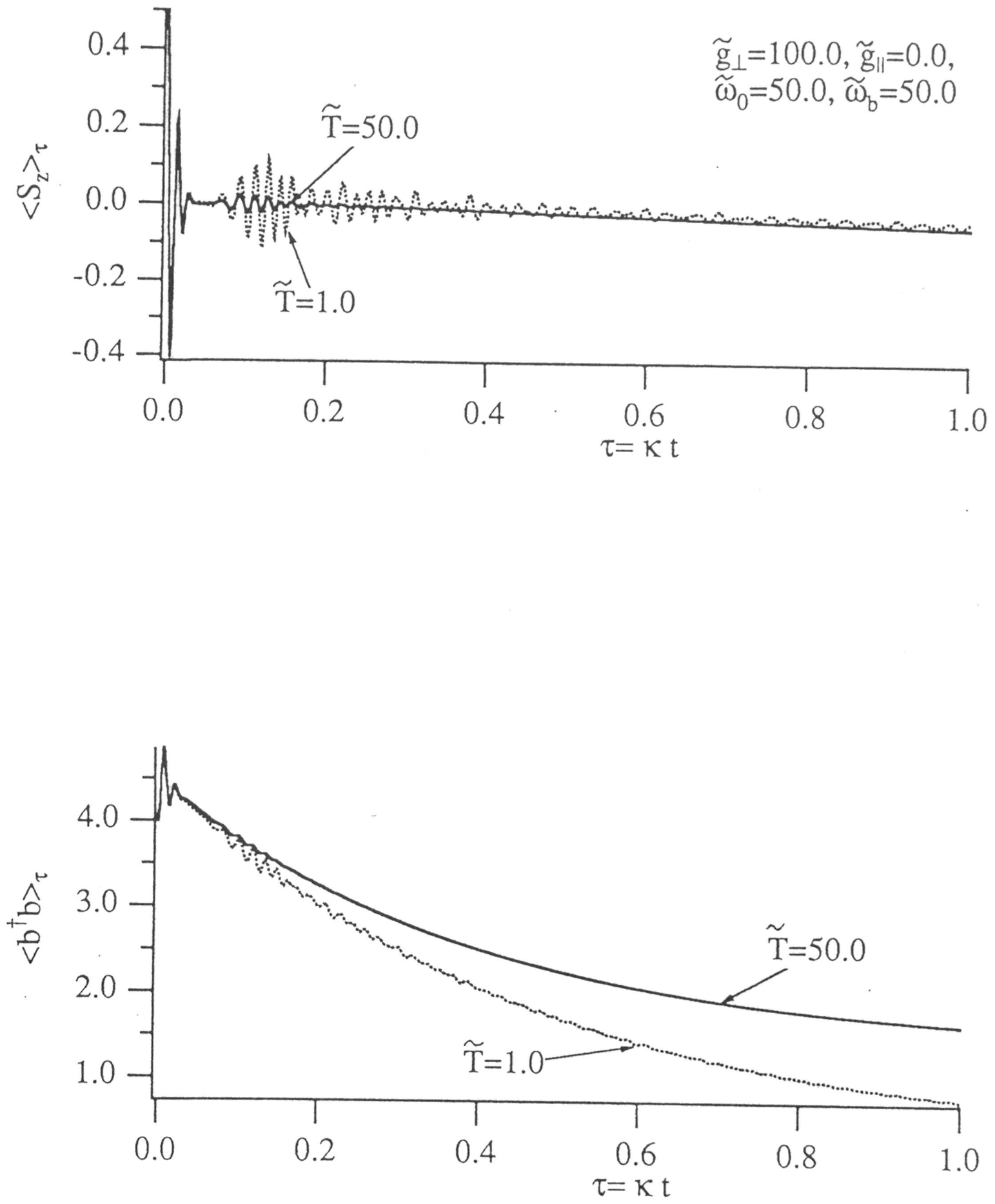
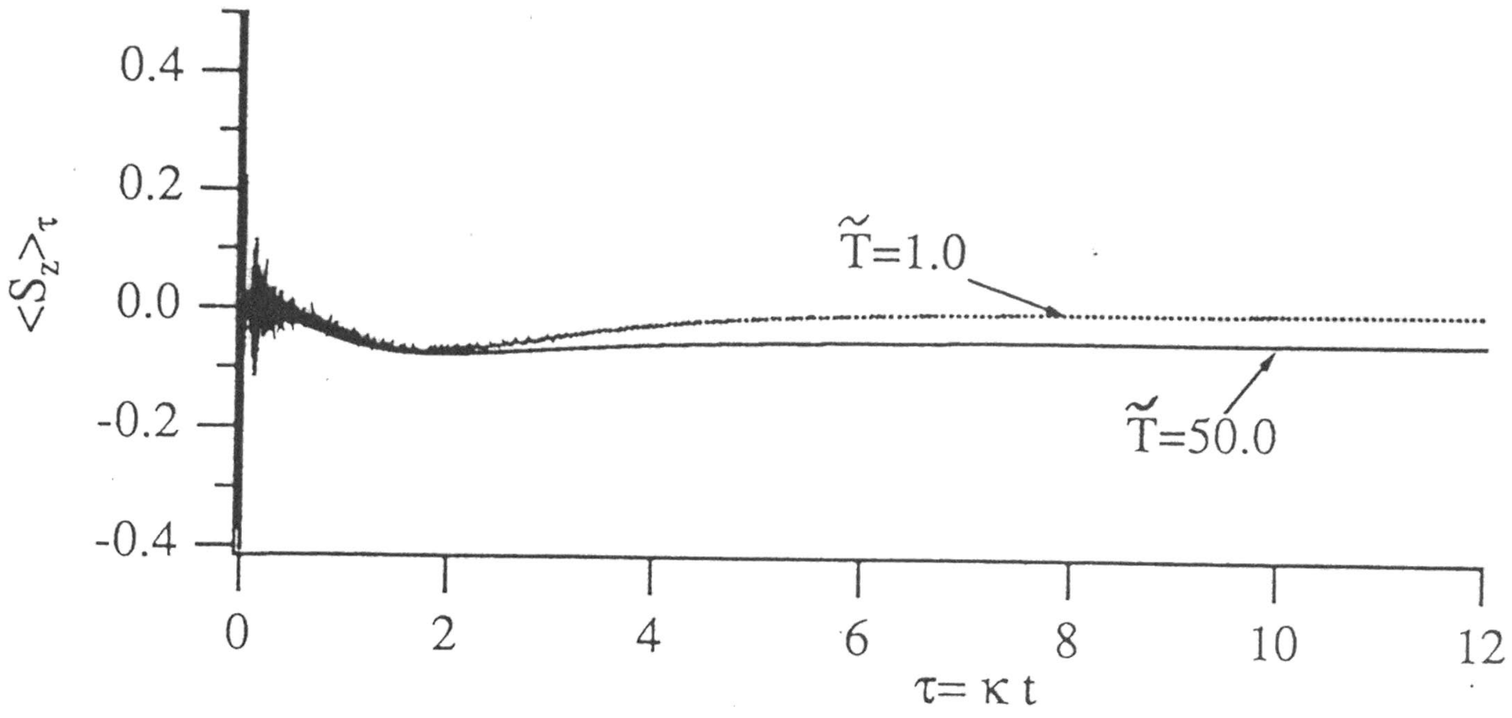


Fig.3.4-a Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the short time regime for the initial condition of $z = 2.0$. System parameters are $\tilde{g}_\perp = g_\perp/\kappa = 100.0$, $\tilde{g}_{||} = g_{||}/\kappa = 0$, $\tilde{\omega}_0 = \omega_0/\kappa = 50.0$, $\tilde{\omega}_b = \omega_b/\kappa = 50.0$ and the damping constants are $\kappa_S = 0$, $\kappa_b = 1.0$ and $\kappa_i = 0$; $\tilde{T} = 1.0$ (dashed line) and $\tilde{T} = 200.0$ (solid line).



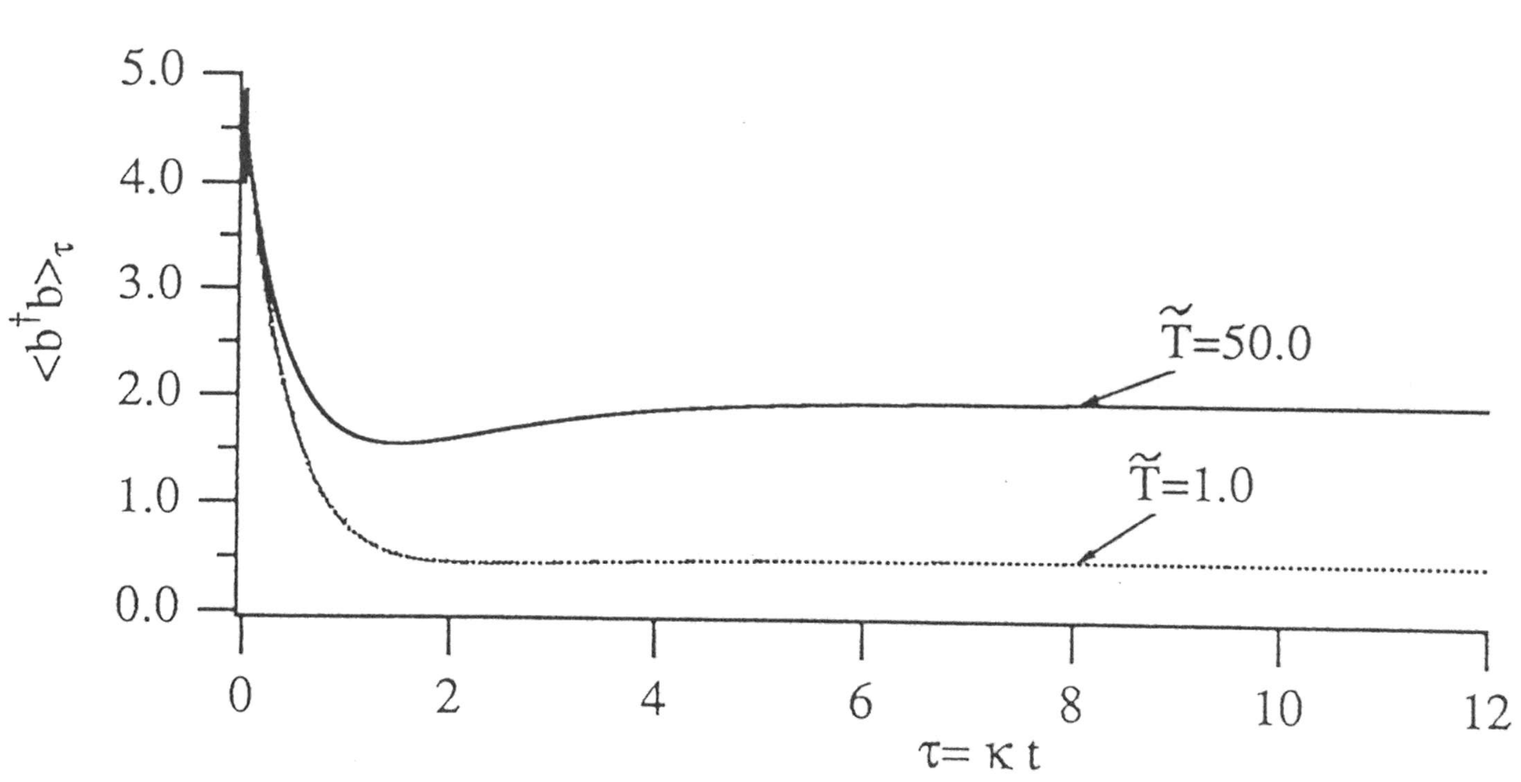
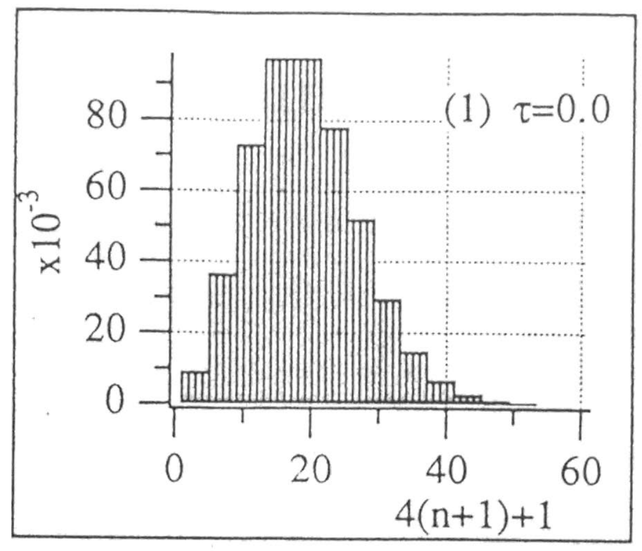
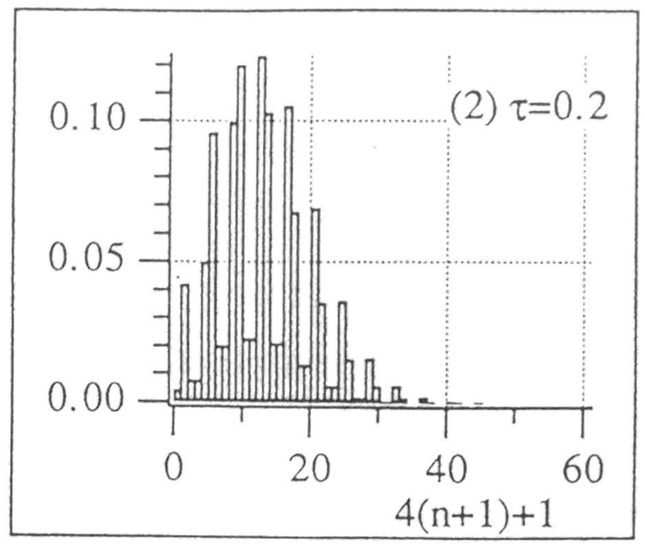
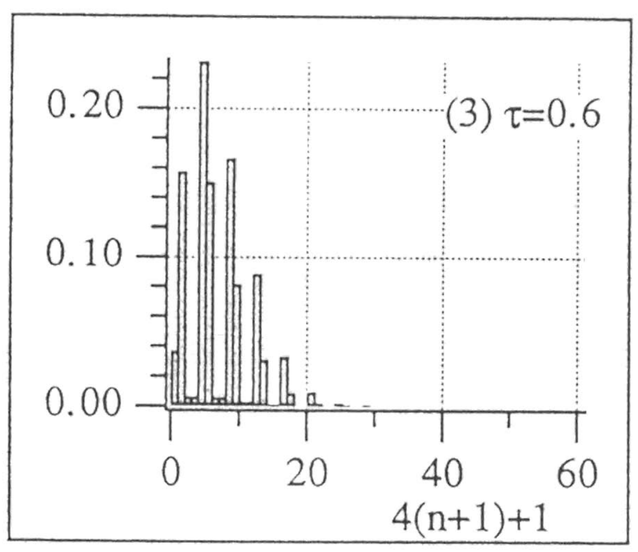
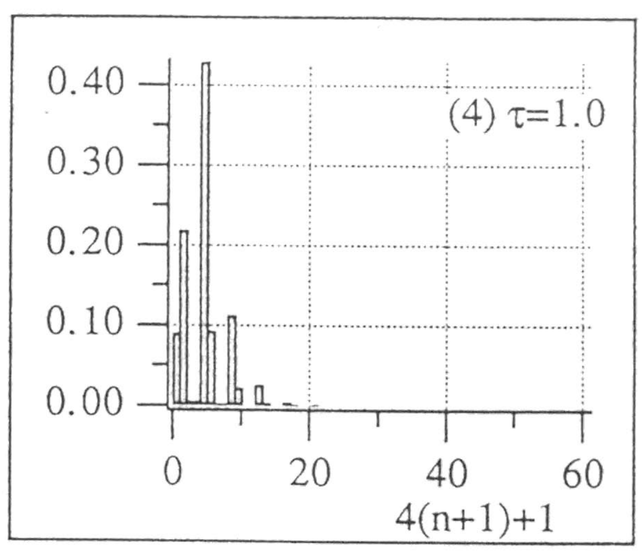


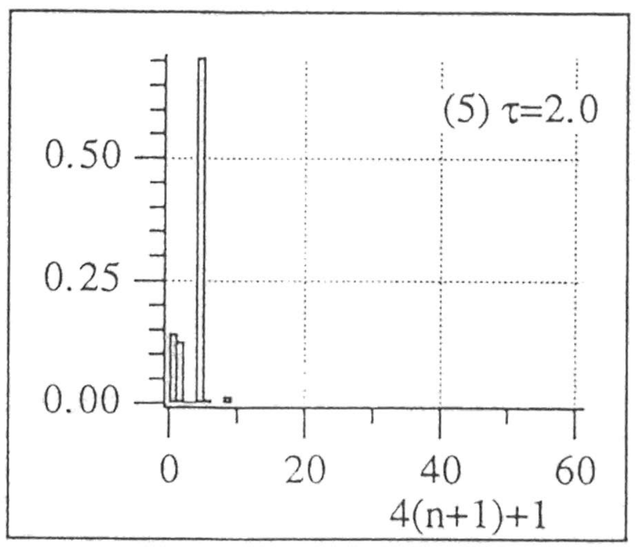
Fig.3.4-b Time evolution of the expectation values $\langle S_z \rangle_\tau$ and $\langle b^\dagger b \rangle_\tau$ in the long time regime.











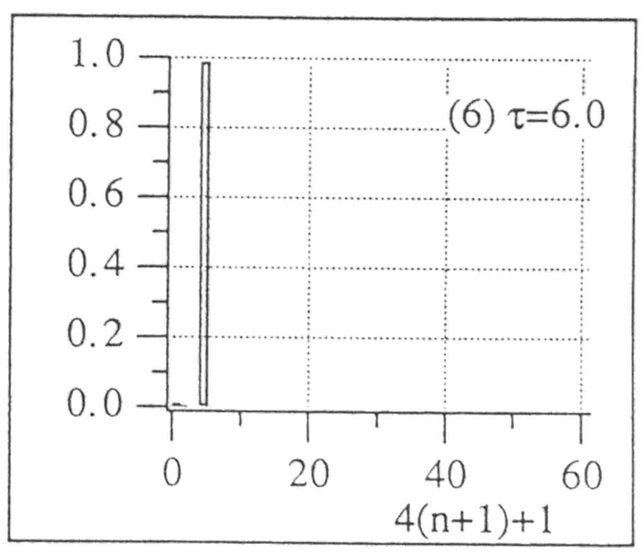


Fig.3.4-c Histograms representation of the reduced density matrix for several values of $\tau = \kappa t$; $T = 1.0$.

3.7 Short summary

We have investigated quantum dynamics of a strongly coupled dissipative system toward the thermal equilibrium with use of a dissipative Jaynes-Cummings model. This model of the coupled system ensures the correct canonical distribution in thermal equilibrium irrespective of the interaction strength between the atom and the field.

We expand the quantal master equation based on TCL formalism in terms of the eigenstates of the Hamiltonian of the coupled system. Time evolution of the elements of the reduced density matrix is described by the vector tri-diagonal differential equation which has the analytic solution.

We consider the relaxation process revealed through the dynamics of elements of the reduced density matrix. Since we expand the density matrix in terms of the eigenstates of the system Hamiltonian, the physical picture of the dynamics of the elements is clear: Decay of the off-diagonal elements (decoherence process) represents the phase relaxation and transitions among the diagonal elements to construct the canonical distribution (diagonal process) represents the energy relaxation. We also show how the expectation values of the physical quantity are related with the matrix elements of the reduced density matrix.

It should be noted that Reynaud and Cohen-Tannoudji [14] used the dressed atom bases to expand the density matrix as has been done in this paper. They treated the resonance fluorescence including collisional effects. Our formalism treats the dissipation effects from the more microscopic point of view, whereas in their work, the collisional master equation is phenomenologically introduced. In our theory, the relaxation times T_1 and T_2, for instance, are obtained from the microscopic calculations. It is interesting to study their problem with use of our model.

By the numerical calculation of the diagonal part in the boson quantum number, quantum characteristics of the strongly coupled dissipative system are found both in the short time regime and the long time regime. The short time regime is characterized by decoherence process, whereas the long time relaxation process is dominated by the diagonal process.

Appendix

3.A Coefficients matrices of the basic equations

The matrices appearing in the basic equation (3.4.45) are tabulated below:

For $n, m \geq 0$,

$$\mathbf{L}^{nm} = \frac{1}{\hbar} \begin{pmatrix} E_+^n - E_+^m & 0 & 0 & 0 \\ 0 & E_+^n - E_-^m & 0 & 0 \\ 0 & 0 & E_-^n - E_+^m & 0 \\ 0 & 0 & 0 & E_-^n - E_-^m \end{pmatrix}, \tag{3.A.1}$$

$$\mathbf{L}^{n\ -1} = \frac{1}{\hbar} \begin{pmatrix} E_+^n - E_0 & 0 \\ 0 & E_-^n - E_0 \end{pmatrix}, \tag{3.A.2}$$

$$\mathbf{L}^{-1\ m} = \frac{1}{\hbar} \begin{pmatrix} E_0 - E_+^m & 0 \\ 0 & E_0 - E_-^m \end{pmatrix} \tag{3.A.3}$$

and

$$\mathbf{L}^{-1\ -1} = 0\ . \tag{3.A.4}$$

For $n, m \geq 1$,

$$\mathbf{p}_S^{\ nm} = \begin{pmatrix} p_{S\ 1111}(n,m) & p_{S\ 1112}(n,m) & p_{S\ 1211}(n,m) & p_{S\ 1212}(n,m) \\ p_{S\ 1121}(n,m) & p_{S\ 1122}(n,m) & p_{S\ 1221}(n,m) & p_{S\ 1222}(n,m) \\ p_{S\ 2111}(n,m) & p_{S\ 2112}(n,m) & p_{S\ 2211}(n,m) & p_{S\ 2212}(n,m) \\ p_{S\ 2121}(n,m) & p_{S\ 2122}(n,m) & p_{S\ 2221}(n,m) & p_{S\ 2222}(n,m) \end{pmatrix} \tag{3.A.5}$$

where

$$p_{S\ \alpha\beta\alpha'\beta'}(n,m) = \left\{ \hat{n}_- \left(\epsilon_{\alpha\beta}^{n-1} \right) + \hat{n}_- \left(\epsilon_{\alpha'\beta'}^{m-1} \right) \right\}\ f_{\alpha\beta}(n-1)\ f_{\alpha'\beta'}(m-1)\,, \tag{3.A.6}$$

$$\mathbf{p}_S^{\ n0} = \mathbf{p}_S^{\ 0n} = \begin{pmatrix} p_{S\ 1011}(n,0) & p_{S\ 1012}(n,0) \\ p_{S\ 2011}(n,0) & p_{S\ 2012}(n,0) \\ p_{S\ 1021}(n,0) & p_{S\ 1022}(n,0) \\ p_{S\ 2021}(n,0) & p_{S\ 2022}(n,0) \end{pmatrix} \tag{3.A.7}$$

where

$$p_{S\ \alpha 0\alpha' \beta'}(n,0) = \left\{ \hat{n}_{-}(\epsilon_{\alpha 0}) + \hat{n}_{-}\left(\epsilon_{\alpha' \beta'}^{n-1}\right) \right\}\ f_{\alpha 0}\ f_{\alpha' \beta'}(n-1), \tag{3.A.8}$$

$$\mathbf{p}_{S}^{\ 00} = \begin{pmatrix} p_{S\ 1010}(0,0) \\ p_{S\ 1020}(0,0) \\ p_{S\ 2010}(0,0) \\ p_{S\ 2020}(0,0) \end{pmatrix} \tag{3.A.9}$$

where

$$p_{S\ \alpha 0\alpha' 0}(0,0) = \{\hat{n}_{-}(\epsilon_{\alpha 0}) + \hat{n}_{-}(\epsilon_{\alpha' 0})\} f_{\alpha 0}\ f_{\alpha' 0}, \tag{3.A.10}$$

and

$$\mathbf{p}_{S}^{\ -1\ -1} = \mathbf{p}_{S}^{\ n\ -1} = \mathbf{p}_{S}^{\ -1\ m} = 0. \tag{3.A.11}$$

For $n, m \geq 0$,

$$\mathbf{q}_{S}^{\ nm} = \begin{pmatrix} q_{S1}(n,m) & q_{S2}(m) & q_{S2}(n) & 0 \\ q_{S3}(m) & q_{S4}(n,m) & 0 & q_{S2}(n) \\ q_{S3}(n) & 0 & q_{S4}(m,n) & q_{S2}(m) \\ 0 & q_{S3}(n) & q_{S3}(m) & q_{S5}(n,m) \end{pmatrix} \tag{3.A.12}$$

where

$$\begin{aligned} q_{S1}(n,m) &= \hat{n}_{-}(\epsilon_{11}^{n})\ \{f_{11}(n)\}^{2} + \hat{n}_{-}(\epsilon_{21}^{n})\ \{f_{21}(n)\}^{2} \\ &+ \hat{n}_{-}(\epsilon_{11}^{m})\ \{f_{11}(m)\}^{2} + \hat{n}_{-}(\epsilon_{21}^{m})\ \{f_{21}(m)\}^{2} \\ &+ \hat{n}_{+}\left(\epsilon_{11}^{n-1}\right)\ \{f_{11}(n-1)\}^{2} + \hat{n}_{+}\left(\epsilon_{12}^{n-1}\right)\ \{f_{12}(n-1)\}^{2} \\ &+ \hat{n}_{+}\left(\epsilon_{11}^{m-1}\right)\ \{f_{11}(m-1)\}^{2} + \hat{n}_{+}\left(\epsilon_{12}^{m-1}\right)\ \{f_{12}(m-1)\}^{2}, \end{aligned} \tag{3.A.13}$$

$$\begin{aligned} q_{S2}(n) &= \hat{n}_{-}(\epsilon_{12}^{n})\ f_{11}(n)\ f_{12}(n) + \hat{n}_{-}(\epsilon_{22}^{n})\ f_{21}(n)\ f_{22}(n) \\ &+ \hat{n}_{+}\left(\epsilon_{21}^{n-1}\right)\ f_{11}(n-1)\ f_{21}(n-1) \\ &+ \hat{n}_{+}\left(\epsilon_{22}^{n-1}\right)\ f_{12}(n-1)\ f_{22}(n-1), \end{aligned} \tag{3.A.14}$$

$$\begin{aligned} q_{S3}(n) &= \hat{n}_{-}(\epsilon_{11}^{n})\ f_{11}(n)\ f_{12}(n) + \hat{n}_{-}(\epsilon_{21}^{n})\ f_{21}(n)\ f_{22}(n) \\ &+ \hat{n}_{+}\left(\epsilon_{11}^{n-1}\right)\ f_{11}(n-1)\ f_{21}(n-1) \\ &+ \hat{n}_{+}\left(\epsilon_{12}^{n-1}\right)\ f_{12}(n-1)\ f_{22}(n-1), \end{aligned} \tag{3.A.15}$$

$$q_{S4}(n,m) = \hat{n}_{-}(\epsilon_{11}^{n})\ \{f_{11}(n)\}^{2} + \hat{n}_{-}(\epsilon_{21}^{n})\ \{f_{21}(n)\}^{2}$$

$$
\begin{aligned}
&+ \quad \hat{n}_{-}\left(\epsilon_{12}^{m}\right)\ \left\{f_{12}\left(m\right)\right\}^{2}+\hat{n}_{-}\left(\epsilon_{22}^{m}\right)\ \left\{f_{22}\left(m\right)\right\}^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{11}^{n-1}\right)\ \left\{f_{11}\left(n-1\right)\right\}^{2}+\hat{n}_{+}\left(\epsilon_{12}^{n-1}\right)\ \left\{f_{12}\left(n-1\right)\right\}^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{21}^{m-1}\right)\ \left\{f_{21}\left(m-1\right)\right\}^{2}+\hat{n}_{+}\left(\epsilon_{22}^{m-1}\right)\ \left\{f_{22}\left(m-1\right)\right\}^{2}, \quad (3.A.16)
\end{aligned}
$$

$$
\begin{aligned}
q_{S5}\left(n,m\right) &= \hat{n}_{-}\left(\epsilon_{12}^{n}\right)\ \left\{f_{12}\left(n\right)\right\}^{2}+\hat{n}_{-}\left(\epsilon_{22}^{n}\right)\ \left\{f_{22}\left(n\right)\right\}^{2} \\
&+ \quad \hat{n}_{-}\left(\epsilon_{12}^{m}\right)\ \left\{f_{12}\left(m\right)\right\}^{2}+\hat{n}_{-}\left(\epsilon_{22}^{m}\right)\ \left\{f_{22}\left(m\right)\right\}^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{21}^{n-1}\right)\ \left\{f_{21}\left(n-1\right)\right\}^{2}+\hat{n}_{+}\left(\epsilon_{22}^{n-1}\right)\ \left\{f_{22}\left(n-1\right)\right\}^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{21}^{m-1}\right)\ \left\{f_{21}\left(m-1\right)\right\}^{2}+\hat{n}_{+}\left(\epsilon_{22}^{m-1}\right)\ \left\{f_{22}\left(m-1\right)\right\}^{2}, \quad (3.A.17)
\end{aligned}
$$

with the following definition

$$
\hat{n}_{+}\left(\epsilon_{11}^{-1}\right)\ \left\{f_{11}\left(-1\right)\right\}^{2}+\hat{n}_{+}\left(\epsilon_{12}^{-1}\right)\ \left\{f_{12}\left(-1\right)\right\}^{2}\equiv\hat{n}_{+}\left(\epsilon_{10}\right)\ \left(f_{10}\right)^{2}, \quad (3.A.18)
$$

$$
\hat{n}_{+}\left(\epsilon_{21}^{-1}\right)\ f_{11}\left(-1\right)\ f_{21}\left(-1\right)+\hat{n}_{+}\left(\epsilon_{22}^{-1}\right)\ f_{12}\left(-1\right)\ f_{22}\left(-1\right)\equiv\hat{n}_{+}\left(\epsilon_{20}\right)\ f_{10}\ f_{20}, \quad (3.A.19)
$$

$$
\hat{n}_{+}\left(\epsilon_{11}^{-1}\right)\ f_{11}\left(-1\right)\ f_{21}\left(-1\right)+\hat{n}_{+}\left(\epsilon_{12}^{-1}\right)\ f_{12}\left(-1\right)\ f_{22}\left(-1\right)\equiv\hat{n}_{+}\left(\epsilon_{10}\right)\ f_{10}\ f_{20}, \quad (3.A.20)
$$

$$
\hat{n}_{+}\left(\epsilon_{21}^{-1}\right)\ \left\{f_{21}\left(-1\right)\right\}^{2}+\hat{n}_{+}\left(\epsilon_{22}^{-1}\right)\ \left\{f_{22}\left(-1\right)\right\}^{2}\equiv\hat{n}_{+}\left(\epsilon_{20}\right)\ \left(f_{20}\right)^{2}, \quad (3.A.21)
$$

and

$$
\mathbf{q}_{S}^{\ n\ -1}=\mathbf{q}_{S}^{\ -1\ n}=\begin{pmatrix} q_{S1}\left(n,-1\right) & q_{S2}\left(n,-1\right) \\ q_{S3}\left(n,-1\right) & q_{S4}\left(n,-1\right) \end{pmatrix} \quad (3.A.22)
$$

where

$$
\begin{aligned}
q_{S1}\left(n,-1\right) &= \hat{n}_{-}\left(\epsilon_{11}^{n}\right)\ \left\{f_{11}\left(n\right)\right\}^{2} \\
&+ \quad \hat{n}_{-}\left(\epsilon_{21}^{n}\right)\ \left\{f_{21}\left(n\right)\right\}^{2} \\
&+ \quad \hat{n}_{-}\left(\epsilon_{10}\right)\ \left(f_{10}\right)^{2}+\hat{n}_{-}\left(\epsilon_{20}\right)\ \left(f_{20}\right)^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{11}^{n-1}\right)\ \left\{f_{11}\left(n-1\right)\right\}^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{12}^{n-1}\right)\ \left\{f_{12}\left(n-1\right)\right\}^{2} \\
&+ \quad \hat{n}_{+}\left(\epsilon_{10}\right)\ \left(f_{10}\right)^{2}, \quad (3.A.23)
\end{aligned}
$$

$$
\begin{aligned}
q_{S2}(n,-1) &= \hat{n}_-(\epsilon_{12}^n)\ f_{11}(n)\ f_{12}(n) \\
&+ \hat{n}_-(\epsilon_{22}^n)\ f_{21}(n)\ f_{22}(n) \\
&+ \hat{n}_+\left(\epsilon_{21}^{n-1}\right)\ f_{11}(n-1)\ f_{21}(n-1) \\
&+ \hat{n}_+\left(\epsilon_{22}^{n-1}\right)\ f_{12}(n-1)\ f_{22}(n-1) \\
&+ \hat{n}_+(\epsilon_{20})\ f_{10}\ f_{20}, \qquad (3.A.24) \\
q_{S3}(n,-1) &= \hat{n}_-(\epsilon_{11}^n)\ f_{11}(n)\ f_{12}(n) \\
&+ \hat{n}_-(\epsilon_{21}^n)\ f_{21}(n)\ f_{22}(n) \\
&+ \hat{n}_+\left(\epsilon_{11}^{n-1}\right)\ f_{11}(n-1)\ f_{21}(n-1) \\
&+ \hat{n}_+\left(\epsilon_{12}^{n-1}\right)\ f_{12}(n-1)\ f_{22}(n-1) \\
&+ \hat{n}_+(\epsilon_{10})\ f_{10}\ f_{20}, \qquad (3.A.25) \\
q_{S4}(n,-1) &= \hat{n}_-(\epsilon_{12}^n)\ \{f_{12}(n)\}^2 \\
&+ \hat{n}_-(\epsilon_{22}^n)\ \{f_{22}(n)\}^2 \\
&+ \hat{n}_-(\epsilon_{10})\ (f_{10})^2 + \hat{n}_-(\epsilon_{20})\ (f_{20})^2 \\
&+ \hat{n}_+\left(\epsilon_{21}^{n-1}\right)\ \{f_{21}(n-1)\}^2 \\
&+ \hat{n}_+\left(\epsilon_{22}^{n-1}\right)\ \{f_{22}(n-1)\}^2 \\
&+ \hat{n}_+(\epsilon_{20})\ (f_{20})^2, \qquad (3.A.26)
\end{aligned}
$$

$$
\mathbf{q}_S{}^{-1\ -1} = 2\ \left\{\hat{n}_-(\epsilon_{10})\ (f_{10})^2 + \hat{n}_-(\epsilon_{20})\ (f_{20})^2\right\}. \qquad (3.A.27)
$$

For $n, m \geq 0$,

$$
\mathbf{q}_i{}^{nm} = \begin{pmatrix}
q_{i1}(n,m) & q_{i2}(m,n) & q_{i2}(n,m) & q_{i3}(n,m) \\
q_{i4}(m,n) & q_{i5}(n,m) & q_{i6}(n,m) & q_{i7}(n,m) \\
q_{i4}(n,m) & q_{i6}(m,n) & q_{i5}(m,n) & q_{i7}(m,n) \\
q_{i8}(n,m) & q_{i9}(n,m) & q_{i9}(m,n) & q_{i10}(n,m)
\end{pmatrix} \qquad (3.A.28)
$$

where

$$
\begin{aligned}
q_{i1}(n,m) &= \hat{n}_+(\delta^n)\left[\{h_{12}(n)\}^2 + \{h_{21}(n)\}^2\right] \\
&+ \hat{n}_+(\delta^m)\left[\{h_{12}(m)\}^2 + \{h_{21}(m)\}^2\right], \qquad (3.A.29) \\
q_{i2}(n,m) &= \hat{n}_-(\delta^n)\left\{h_{11}(n) - h_{11}(m)\right\}\left\{h_{12}(n) + h_{21}(n)\right\}, \qquad (3.A.30)
\end{aligned}
$$

$$q_{i3}(n,m) = -\left\{\hat{n}_-(\delta^n)+\hat{n}_-(\delta^m)\right\}\left\{h_{12}(n)\,h_{12}(m)+h_{21}(n)\,h_{21}(m)\right\}, \tag{3.A.31}$$

$$q_{i4}(n,m) = -\hat{n}_+(\delta^n)\left\{h_{11}(n)+h_{11}(m)\right\}\left\{h_{12}(n)+h_{21}(n)\right\}, \tag{3.A.32}$$

$$\begin{aligned} q_{i5}(n,m) &= \hat{n}_+(\delta^n)\left[\left\{h_{12}(n)\right\}^2+\left\{h_{21}(n)\right\}^2\right] \\ &+ \hat{n}_-(\delta^m)\left[\left\{h_{12}(m)\right\}^2+\left\{h_{21}(m)\right\}^2\right], \end{aligned} \tag{3.A.33}$$

$$q_{i6}(n,m) = -\left\{\hat{n}_-(\delta^n)+\hat{n}_+(\delta^m)\right\}\left\{h_{12}(n)\,h_{21}(m)+h_{21}(n)\,h_{12}(m)\right\}, \tag{3.A.34}$$

$$q_{i7}(n,m) = \hat{n}_-(\delta^n)\left\{h_{12}(n)+h_{21}(n)\right\}\left\{h_{11}(n)+h_{11}(m)\right\}, \tag{3.A.35}$$

$$q_{i8}(n,m) = -\left\{\hat{n}_+(\delta^n)+\hat{n}_+(\delta^m)\right\}\left\{h_{12}(n)\,h_{12}(m)+h_{21}(n)\,h_{21}(m)\right\}, \tag{3.A.36}$$

$$q_{i9}(n,m) = -\hat{n}_+(\delta^n)\left\{h_{12}(n)+h_{21}(n)\right\}\left\{h_{11}(n)-h_{11}(m)\right\}, \tag{3.A.37}$$

$$\begin{aligned} q_{i10}(n,m) &= \hat{n}_-(\delta^n)\left[\left\{h_{12}(n)\right\}^2+\left\{h_{21}(n)\right\}^2\right] \\ &+ \hat{n}_-(\delta^m)\left[\left\{h_{12}(m)\right\}^2+\left\{h_{21}(m)\right\}^2\right], \end{aligned} \tag{3.A.38}$$

with $\delta^n = 2\lambda(n)$,

$$\mathbf{q}_i{}^{n\ -1} = \mathbf{q}_i{}^{-1\ n} = \begin{pmatrix} q_{i1}(n,-1) & q_{i2}(n,-1) \\ q_{i3}(n,-1) & q_{i4}(n,-1) \end{pmatrix} \tag{3.A.39}$$

where

$$q_{i1}(n,-1) = \hat{n}_+(\delta^n)\left[\left\{h_{12}(n)\right\}^2+\left\{h_{21}(n)\right\}^2\right], \tag{3.A.40}$$

$$q_{i2}(n,-1) = \hat{n}_-(\delta^n)\,h_{11}(n)\left\{h_{12}(n)+h_{21}(n)\right\}, \tag{3.A.41}$$

$$q_{i3}(n,-1) = -\hat{n}_+(\delta^n)\,h_{11}(n)\left\{h_{12}(n)+h_{21}(n)\right\}, \tag{3.A.42}$$

$$q_{i4}(n,-1) = \hat{n}_-(\delta^n)\left[\left\{h_{12}(n)\right\}^2+\left\{h_{21}(n)\right\}^2\right], \tag{3.A.43}$$

and

$$\mathbf{q}_i{}^{-1\ -1} = 0. \tag{3.A.44}$$

For $n, m \geq 0$,

$$\mathbf{r}_S{}^{nm} = \begin{pmatrix} r_{S\ 1111}(n,m) & r_{S\ 1121}(n,m) & r_{S\ 2111}(n,m) & r_{S\ 2121}(n,m) \\ r_{S\ 1112}(n,m) & r_{S\ 1122}(n,m) & r_{S\ 2112}(n,m) & r_{S\ 2122}(n,m) \\ r_{S\ 1211}(n,m) & r_{S\ 1221}(n,m) & r_{S\ 2211}(n,m) & r_{S\ 2221}(n,m) \\ r_{S\ 1212}(n,m) & r_{S\ 1222}(n,m) & r_{S\ 2212}(n,m) & r_{S\ 2222}(n,m) \end{pmatrix} \tag{3.A.45}$$

where

$$r_{S\ \alpha\beta\alpha'\beta'}(n,m) = \left\{\hat{n}_+\left(\epsilon^n_{\alpha\beta}\right) + \hat{n}_+\left(\epsilon^m_{\alpha'\beta'}\right)\right\}\ f_{\alpha'\beta}(n)\ f_{\alpha'\beta'}(m), \tag{3.A.46}$$

$$\mathbf{r}_S{}^{n\ -1} = \mathbf{r}_S{}^{-1\ n} = \begin{pmatrix} r_{S\ 1011}(n,-1) & r_{S\ 2011}(n,-1) & r_{S\ 1021}(n,-1) & r_{S\ 2021}(n,-1) \\ r_{S\ 1012}(n,-1) & r_{S\ 2012}(n,-1) & r_{S\ 1022}(n,-1) & r_{S\ 2022}(n,-1) \end{pmatrix} \tag{3.A.47}$$

where

$$r_{S\ \alpha0\alpha'\beta'}(n,-1) = \left\{\hat{n}_+(\epsilon_{\alpha0}) + \hat{n}_+\left(\epsilon^n_{\alpha'\beta'}\right)\right\}\ f_{\alpha0}\ f_{\alpha'\beta'}(n) \tag{3.A.48}$$

and

$$\mathbf{r}_S{}^{-1\ -1} = \begin{pmatrix} r_{S\ 1010}(-1,-1) & r_{S\ 1020}(-1,-1) & r_{S\ 1020}(-1,-1) & r_{S\ 2020}(-1,-1) \end{pmatrix} \tag{3.A.49}$$

where

$$r_{S\ \alpha0\alpha'0}(-1,-1) = \{\hat{n}_+(\epsilon_{\alpha0}) + \hat{n}_+(\epsilon_{\alpha'0})\}\ f_{\alpha0}\ f_{\alpha'0}. \tag{3.A.50}$$

The coefficient matrices $\mathbf{p}_b{}^{nm}$, $\mathbf{q}_b{}^{nm}$ and $\mathbf{r}_b{}^{nm}$ are obtained by replacing (3.4.12)-(3.4.17) in $\mathbf{p}_S{}^{nm}$, $\mathbf{q}_S{}^{nm}$ and $\mathbf{r}_S{}^{nm}$ by (3.4.18)-(3.4.21).

Bibliography

[1] E. T. Jaynes and F. W. Cummings, Proc. IEEE **51** (1963) 89.

[2] See for instance, B. W. Shore and P. L. Knight, J. Mod. Optics **40** (1993), 1195, and references cited therein.

[3] See for instance, R. J. Brecha, A. Peters, C. Wagner and H. Walter, Phys. Rev. A **46** (1992) 567 and references cited therein.

[4] See for instance, *Cavity Quantum Electrodynamics*, Advances in Atomic, Molecular, and Optical Physics, Supplement 2, ed. P. R. Berman, (Academic Press, Boston, 1994).

[5] See for instance, F. Haake, Springer Tracts in Modern Physics Vol. 66, 98, (Springer-Verlag, Berlin, 1973).

[6] W. E. Louisell, *Quantum Statistical Properties of Radiation*, (John Wiley & Sons, Inc., New York, 1973).

[7] See for instance, R. Kubo, M. Toda and N. Hashitsume, *Statistical Physics II*, 89, (Springer-Verlag, Berlin, 1985).

[8] J. D. Cresser, J. Mod. Optics, **39** (1992), 2187.

[9] M. Murao and F. Shibata, Physica A **216** (1995), 255.

[10] C. Cohen-Tannoudji, in *Frontiers in Laser Spectroscopy*, edited by R. Balian, S. Haroche and S. Liberman, (North Holland, Amsterdam, 1977), Vol.1.

[11] C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg, *Atom-Photon Interactions: basic processes and applications*,
(John Wiley & Sons, Inc., New York, 1992).

[12] F. Shibata and T. Arimitsu, J. Phys. Soc. Jpn. **49** (1980), 891.

[13] See for instance, H. Risken, *The Fokker-Planck Equation*,
(Springer-Verlag, Berlin, 1984).

[14] S. Reynaud and C. Cohen-Tannoudji, J. Physique **43** (1982), 1021.