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WEB GEOMETRY of SOLUTIONS of FIRST ORDER ODEs

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and currently can be found also at

http://citeseerx.ist.psu.edu/viewdoc/download;jsessionid=122F585CD11770DC0A3382F6D7321474?doi=10.1.1.538.8920&rep=rep1&type=pdf

WEB GEOMETRY of SOLUTIONS of FIRST ORDER ODEs

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1 Affine Connection of 3-webs

A first order ordinary differential equation of one valuable (ODE) is

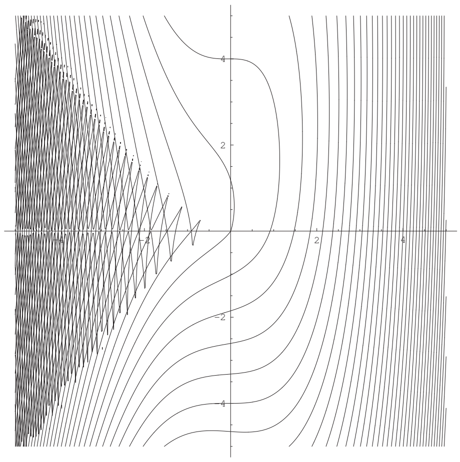
$$f(x, y, y') = 0 \qquad (*)$$

where y' stands for the derivative dy/dx and f is a germ of real or complex analytic function at $(0, 0, p_0)$. In suitable coordinates one may assume $p_0 = 0$. The variety in the xyp-space defined by $f(x, y, p) = 0$ is called the *skeleton* of the equation, on which the natural projection onto the xy-space restricts to a branched covering map. f is *equivalent* to an equation $g(x, y, y') = 0$ if there exists a germ of diffeomorphism ϕ of xy-plane which sends the solutions of f to those of g, in other words, the lift $d\phi$ of ϕ sends the skeleton of f to that of g.

In order to observe the solutions of f, one may solve the equation in y' locally as

$$y' = f_i(x, y) \ , \ i = 1, \ldots, d$$

using implicit functions f_i. The solutions of each explicit differential equation form a germ of foliation by curves, hence the solutions of the equation $(*)$ form a configuration of d foliations. Such a structure is called a *d-WEB* of codimension 1, or simply *WEB*, and has been long studied by differential geometers such as Cartan, Blaschke (see c.f. [1,2,3]).



Non symmetric Wave front 3-web

The above figure shows a tipical singular

3-web structure, where $p_0 = \infty$. This is a generic member in a versal deformation family of infinite dimension of the equation of the normal form

$$y'^3 + yy' - x = 0 \quad \text{(symmetric wave front)}$$

(For the notion of the versal deformation family, see [5].) All these differential equations may be defined in terms of the stationary phase nethod. The critical values of the potential function

$$V(p) = \frac{1}{4}p^4 + \frac{1}{2}xp^2 + yp + h(x, y)$$

with an analytic constant term h is a 3-valued analytic function of x, y. The level sets of this function are analytic curves interweaved together and constitute the set of solutions of a first order (implicit) differential equation. Regarding the critical value as a function of x, y and the free valuable h, the level sets define a singular 3-web on the xyh-space, which is called the versal web. The above portrait is nothing but a cross section of the versal web. The minimal system of partial differential equations which the level sets in xyh-space satisfy is the *versal* PDE unfolding these ODEs on the plane. The existense of such versal PDE was proved for some cases where the skeleton is smooth. In such cases the classification of ODE falls into that of the cross section.

One of basic ideas to extract geometric invariants is to extend Bott connection of these foliations (if possible) to an equal affine connection ∇ of the xy-space. In the case $d = 3$, such a connection is called *Chern connection.* This connection is defined on the complement of the discriminant of the equation, and extends meromorphically to the discriminant ([4]). The singuarity of the connection depends subtly on that of the equation in general. So one may expect to classify the equations in terms of affine connectoin, and in some case only by their curvature forms.

In order to introduce such an affine connection let

$$\omega_i = U_i\ (dy - f_i\ dx), \quad i = 1, \ldots, d$$

with functions $U_i \neq 0$. In the simplest non trivial case of $d = 3$, one may impose the *normalization condition*

$$\omega_1 + \omega_2 + \omega_3 = 0 \quad .$$

Then it is seen that there exists a unique θ such that

$$d \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix} \wedge \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} + (T = 0) \quad (**)$$

The i-th row of the equation

$$d\omega_i = \theta \wedge \omega_i$$

is nothing but the integrability condition of ω_i. Forgetting the i-th row this equation defines an affine connection without torsion which we denote by ∇. By the above normalization condition, the resulting connection is independent of i, which is the *Chern connection* of the 3-web of $\omega_1, \omega_2, \omega_3$. It has the connection form $\Theta = \theta I$ and the curvature form

$$\Omega = d\Theta + \Theta \wedge \Theta = d\theta\ I$$

where I stands for the 2×2 identity matrix. The curvature form is independent of the choice of co-frames since it is a similarity matrix.

Let S denote the skeleton in the xyp-space (i.e. the 1-jet space $J^1(R,R)$) defined by $f(x,y,p) = 0$. The S is locally identified with the xy-plane via the natural projection. The above method is generalized to define an affine connection on (the smooth part of) S, which is an extention of the Bott connection of the foliation on S defined by the contact form.

Here we present a small local normal form theorem.

Theorem 1.1 (Lins Neto, Nakai[4]) Assume the natural projection of the skeleton $f(x,y,p) = 0$ to xy-plane has multiplicity 3. Then f is equivalent to the polynomial equation of degree 3

$$y'^3 + B\, y' + C = 0 \qquad (***)$$

where B, C are germs of analytic functions.

Mignard[7] calculated the curvature form of normal form $(***)$ and obtained a fuge formula using computer. Henaut[6] arrived at the formula in a better presentation showing an insight from the D-module theory. Below we present the curvature form in the simplest form.

Theorem 1.2 (Mignard[7], Henaut[6], Lins Neto, Nakai[4]) The $d\theta$ in the curvature form of the normal form $(***)$ is the sum

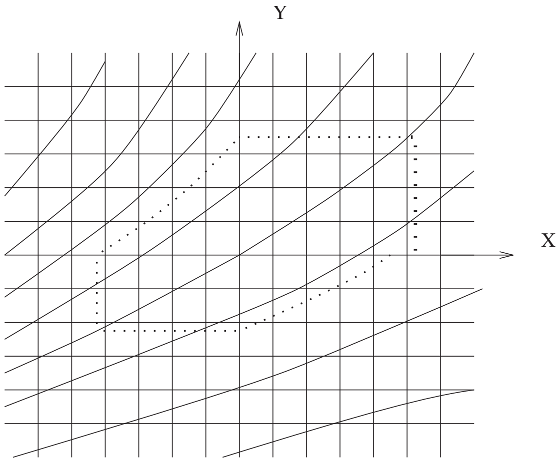
$$\frac{1}{6}\,(\log\Delta)_{xy}\; dx \wedge dy$$
$$+ \text{ the exterior differentiation of}$$
$$\frac{(6BB_yC - 4B^2C_y)\;dx + (6BC_x - 9B_xC)\;dy}{\Delta}$$

where $\Delta = 4B^3 + 27C^2$ is the discriminant of the cubic polynomial in p.

Theorem 1.3 (Resonant Curve Theorem, Lins Neto,Nakai[4]) Assume $d = 3$ and the the germ of discriminant Δ at the origin is diffeomorphic to the $(2,3)$ cusp. Then the curvature form has pole of order 2 on the discriminant and vanishes on a union of 2 non singular curves passing through the origin: one is tangent to the discriminant at the origin and the other is transverse.

2 The flat 3-web and non flat 3-webs

A *non singular 3-web* of the plane is a configuration of 3 non singular curvilinear foliations, which are mutually in general position. Portrait of such a generic 3-web is as follows.



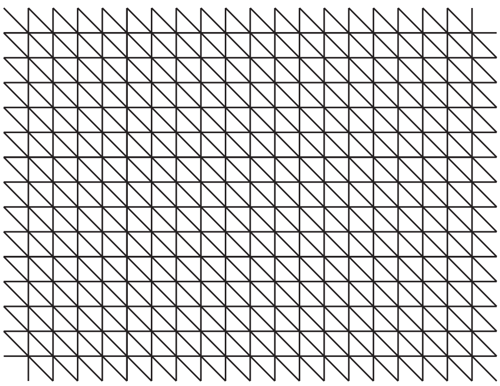
A non hexagonal 3-web

A non singular 3-web is *flat*, or in other words, *hexagonal* if its Chern connection is flat, in other words, the curvature 2-form $d\theta$ is identically 0. The following fact is classically known (see c.f. [3]).

Theorem 2.1 (Linearlization Theorem [1,2,4]) *A flat 3-web is locally diffeomorphic to the linear 3-web defined by*

$$dx \ , \ dy \ , \ -(dx + dy) \ .$$

The following figure shows the linear 3-web structure.



The linear 3-web

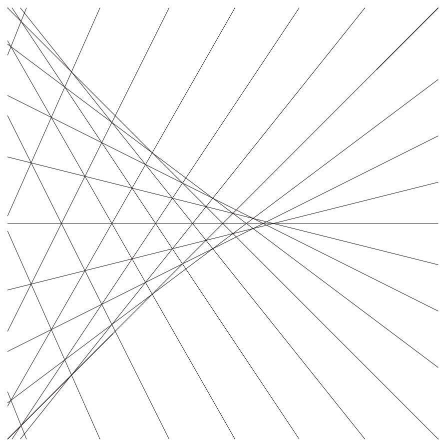
3 Flat Differential equations

Theorem 3.1 (Lins Neto, Nakai[4]) *Assume the 3-web defined by f is flat and the discriminant is diffeomorphic to the $(2,3)$ cusp. Then f is equivalent to one of the following 2 equations.*

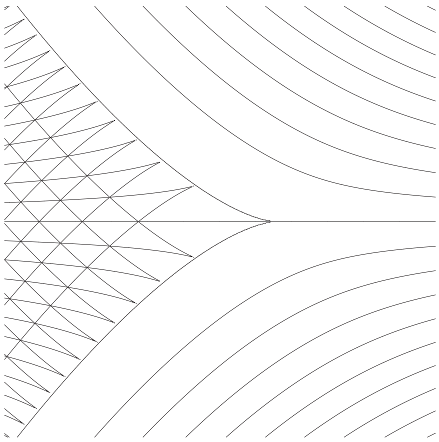
$$y'^3 + xy' - y = 0, \tag{1}$$

$$y'^3 + \frac{1}{4}xy' + \frac{1}{8}y = 0. \tag{2}$$

The flat 3-webs defined by these equations are respectively as follows.



(1): Clairaut 3-web



(2): Rectangular 3-web

The reader may observe the affine (linear) structure on the complement of the discriminant.

The 3-web structure can be also seen on the homogeneous Fermat surfase V_α defined by

$$x^\alpha + y^\alpha + z^\alpha = 0 \ .$$

It is seen that the coordinate functions x, y, z cut a flat 3-web on V_α. The quotient of the surface by the permutation of coordinates is also an algebraic surface on which a flat 3-web structure is induced.

Theorem 3.2 *The quotient V_α admits a finite-to-one parametrization by plane if and only if the exponent α is one of the following list.*

$$\pm 1 \ , \ \frac{1}{2} \ , \ \pm\frac{1}{3} \ , \ \frac{1}{6} \ , \ \pm 2 \ , \ \pm\frac{2}{3} \ , \ \pm 3 \ , \ \frac{3}{2} \ ,$$
$$\pm 4 \ , \ \pm\frac{4}{3}, \ \pm 5, \ \frac{5}{2} \ , \ \pm\frac{5}{3} \ , \ \frac{5}{6}$$

And on the parameter space a flat 3-web structure is induced and it defines a flat differential equation.

The equations in the above theorem are not local in p in general and do not fall into the classifacation in Theorem 3.1. "Almost all" flat equations are induced from these equations by finite-to-one mapping.

4 Dual 3-web

The *dual 3-line configuration* of a configuration $L = L_1 \cup L_2 \cup L_3$ of 3 lines in the plane passing through the origin is the unique invariant 3-line configuration (different from L) of the group generated by three involutions respecting the line L_i and L. The *dual 3-web* W^* of a 3-web W is defined by integrating the dual 3-line configuration of the tangent 3-line fields of W.

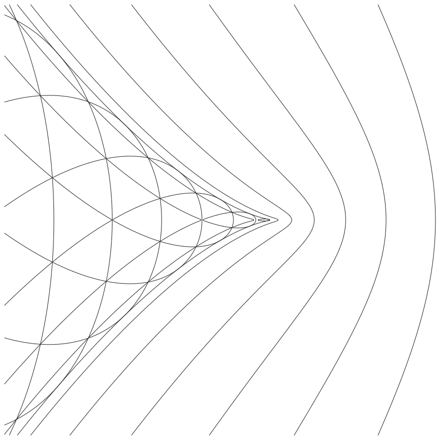
Theorem 4.1 *The bi-duality holds: $W^{**} = W$, and W and W^* have the same Chern connection.*

Corollary 4.1 *A 3-web W is flat if and only if its dual W^* is flat.*

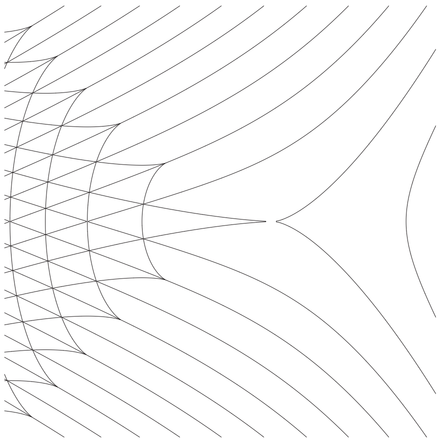
The dual equations of (1), (2) in Theorem 3.1 are respectively

$$y'^3 + \frac{2x^2}{3y}\, y'^2 - x\, y' + \frac{2x^3 + 27y^2}{27y} = 0, \quad (3)$$
$$y'^3 - \frac{x^2}{3y}\, y'^2 - \frac{x}{4}\, y' - \frac{2x^3 + 27y^2}{216y} = 0 \quad (4)$$

The 3-web structure of these equations are as follows.



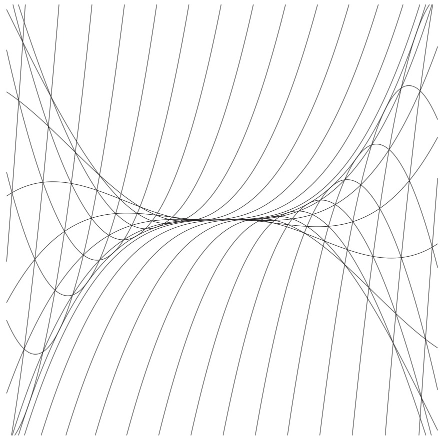
(3): Dual 3-web of (1)



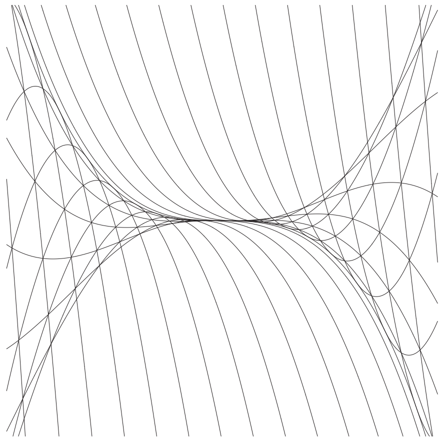
(4): Dual 3-web of (2)

5 A Self-dual flat 3-web

Consider the following flat 3-web obtained by folding the linear 3-web (1) by the antipodal involution of the y-coordinate, $y \to -y$.



The dual of this web is the following



Dual of the Self-dual flat 3-web

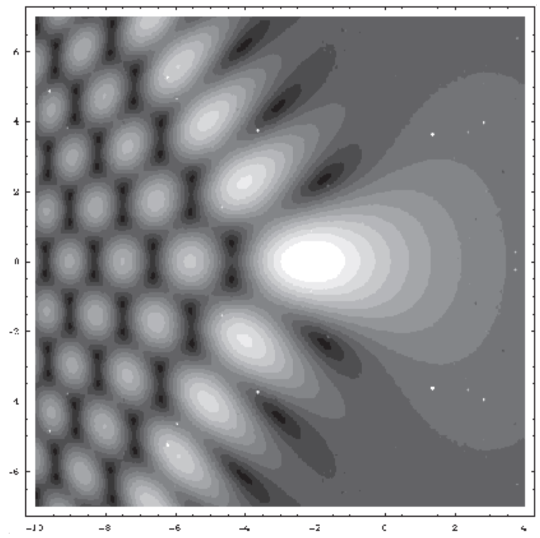
It is senn that the dual web is the rotation of the origina web.

6 Relation to the stationary phase method

Let us onsider the Pearcy integral

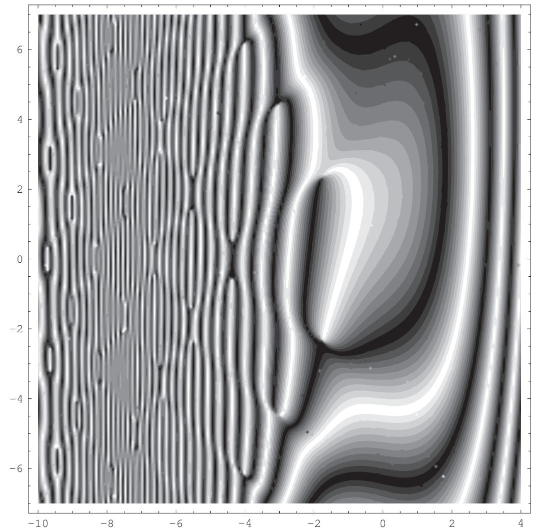
$$\int_{-\infty}^{\infty} \exp \sqrt{-1}\{\frac{1}{4}p^4+\frac{1}{2}xp^2+yp+h(x,y)\}\, dp$$

The absolute value of the integral approximates the intensity of light inside a cuspidal optical caustics. Clearly the intensity is independent of the constant term h. The contour is as follows.



Contour of the intensity

On the other hand the argument of the integral approximates the phase, which depends on the h.



Contour of phase

The portrait in Figure (2) may be defined by level curves of difference $\alpha-\beta, \beta-\gamma, \gamma-\alpha$ of 3 phases α, β, γ.

It would be interesting to compare the above figure to the first figure in this paper, which is defined by the level curves of the 3-valued phase function. According to theorem 1.3, there exist two smooth curves passing through the cusp, on a infinitesimally small neighbourhood of which the 3-web structure is flat, and resonance of 3-waves is observed at discrete points on those curves. The wave in the last figure is moving from the right to the left, but some "trajectories" might be trapped into the eddies inside cusp.

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