Abstract

The geometric structures of the associative Grassmann manifold and their applications

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The algebra $\mathbb{O}$ of octonions is known as a normed algebra which is neither commutative nor associative. A three-dimensional oriented subspace in the subspace $\mathrm{Im}\mathbb{O}$ of all imaginary part of octonions is called an *associative subspace* if it is the canonically oriented imaginary part of some quaternion subalgebra of octonions. The set of all associative subspaces is called the *associative Grassmann manifold* which is denoted by $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. The purpose of this paper is to obtain some new geometric knowledge on $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ by describing in detail its geometric structures, such as its Riemannian symmetric pair and quaternionic Kähler structure.

The exceptional compact Lie group G_2 which is the automorphism group of octonions acts transitively on $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, and the isotropy subgroup of G_2 at the subspace of imaginary quaternions is $SO(4)$. Thus it is known that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is diffeomorphic to a Riemannian symmetric space $G_2/SO(4)$. We also have the fact that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O}) \cong G_2/SO(4)$ is an eight-dimensional compact symmetric quaternionic Kähler manifold. Quaternionic Kähler structure is interesting by its own and has possible applications. The study of submanifold theory of quaternionic Kähler manifold has progressed steadily. In particular, studying complex submanifolds of a quaternionic Kähler manifold is of interest in the field of quaternionic complex differential geometry where complex and quaternionic differential geometries interact. So it is meaningful to describe explicitly the quaternionic Kähler structure of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ and develop its complex submanifold theory.

The Lie group G_2 plays an important role in various fields of geometry. For example, the study of submanifolds characterized by the associative calibration and the coassociative calibration defined on $\mathrm{Im}\mathbb{O}$ has made progress. It is known that a six-dimensional unit sphere in $\mathrm{Im}\mathbb{O}$ admits an almost complex structure induced from octonions and is diffeomorphic to $G_2/SU(3)$. Many researchers have contributed to the study of its almost complex submanifolds and Lagrangian submanifolds. We can say that fruitful achievements are being obtained in the field of G_2-geometry.

We believe that $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is a fascinating subject where the quaternionic differential geometry and the G_2-geometry interact.

This paper has six sections organized as follows:

In Section 1, we recall some basic facts about the algebra of quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ as well as the Lie group G_2. We denote by $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ a Grassmann manifold of all three-dimensional oriented subspaces in $\mathrm{Im}\mathbb{O}$. Then level sets $\tilde{M}(t)$ $(-1 \leq t \leq 1)$ of the associative calibration on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$

are defined. They are the G_2-orbits in $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. Therefore for $-1 < t < 1$, $\tilde{M}(t)$ are homogeneous hypersurfaces in $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$. The level set $\tilde{M}(1)$ coincides with $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$.

Then in Section 2, we describe the Riemannian symmetric pairs of the Grassmann manifolds $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ explicitly. Moreover, we prepare the standard decomposition of the Lie algebra $\mathfrak{g}_2$ of the Lie group G_2 and give its basis. Then we express the tangent space of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ at the base point $\mathrm{Im}\mathbb{H}$ by using matrices.

In Section 3, we introduce the quaternionic Kähler structure of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ in two different ways. First, we describe it by identifying the tangent space of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ at $\mathrm{Im}\mathbb{H}$ with the space of linear homomorphisms of $\mathrm{Im}\mathbb{H}$ to $\mathbb{H}$ which satisfy a certain condition. The second description is to make use of the matrix expression given in Section 2 based on the Lie algebra theory.

In Section 4, we compute the principal curvatures of each $\tilde{M}(t)$ $(-1 < t < 1)$ in $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ using the fact that $\tilde{M}(t)$ are tubular hypersurfaces around $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$. Homogeneous hypersurfaces are given as orbits by the cohomogeneity one actions. A. Kollross classified cohomogeneity one actions on irreducible Riemannian symmetric spaces of compact type. By his classification, it is known that most of the cohomogeneity one actions are the Hermann actions. Then the principal curvatures of homogeneous hypersurfaces by the Hermann actions of cohomogeneity one have been calculated by many researchers. The G_2-action on $\widetilde{\mathrm{Gr}}_3(\mathrm{Im}\mathbb{O})$ is one of the "exceptional" cohomogeneity one actions, that is, non-Hermann type. By making use of the explicit expression of $\mathfrak{g}_2$, we compute the principal curvatures. As applications, we show that there is a unique orbit which is an austere submanifold, and that there are just two orbits which are proper biharmonic homogeneous hypersurfaces. We also show that the austere orbit is a weakly reflective submanifold.

We focus on Lagrangian submanifolds of S^6 and study the relationship of such submanifolds with the geometry of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ in Section 5. First, we review the Lagrangian geometry of S^6. Then the double fibration onto S^6 and $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ is defined. Lagrangian submanifolds of S^6 are associated with the geometry of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ through the double fibration. In particular, we show that the Gauss maps of into $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ associated to Lagrangian submanifolds are harmonic. Then we discuss the Gauss maps associated to homogeneous Lagrangian submanifolds constructed by K. Mashimo.

In Section 6, examples of transversally complex submanifolds and totally complex submanifolds of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$ are obtained. Among compact symmetric quaternionic Kähler manifolds, the study of the quaternionic projective space $\mathbb{H}P^n$, a complex Grassmann manifold $\mathrm{Gr}_2(\mathbb{C}^n)$ and a real Grassmann manifold $\widetilde{\mathrm{Gr}}_4(\mathbb{R}^n)$ has made progress. In particular, $2n$-dimensional totally complex submanifolds which are extrinsically symmetric in $\mathbb{H}P^n$ are classified. Making use of the quaternionic Kähler structure of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$, we give examples of transversally complex submanifolds of $\widetilde{\mathrm{Gr}}_{ass}(\mathrm{Im}\mathbb{O})$.