外国語要約

Representation theory of compact quantum groups based on operator algebras and its application

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1 Compact Quantum Group

A Banach algebra equipped with the $*$-operation satisfying the C*-condition $\|T^*T\| = \|T\|^2$ is called a C*-algebra. One of the most fundamental theorem in the theory of operator algebras is Gelfand–Naimark theorem, which states that a commutative operator algebra (unital C*-algebra) is isomorphic to the algebra of continuous functions on some compact space. According to this fundamental result, a general operator algebra can be seen as an algebra of continuous functions on a hypothetical "noncommutative" compact topological space. The theory of compact quantum groups by Woronowicz [5] provides a group structure on such research objects.

Definition. A compact quantum group is a pair (A, Δ) of an unital C*-algebra A and an unital $*$-homomorphism $\Delta\colon A \to A \otimes A$ called comultiplication such that

1. (coassociativity) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,
2. (cancelation property) the spaces

$$(A \otimes 1)\Delta(A) = \mathrm{span}\{(a \otimes 1)\Delta(b) | a, b \in A\},\ (1 \otimes A)\Delta(A) = \mathrm{span}\{(1 \otimes a)\Delta(b) | a, b \in A\}$$

are dense in $A \otimes A$.

Definition. Let q be a real number such that $|q| \leq 1$, and $q \neq 0$. The quantum $SU(2)$ group $SU_q(2)$ is defined as follows. The algebra $C(SU_q(2))$ is the universal C*-algebra generated by two elements α and γ such that

$$(u^q_{ij})_{i,j} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary.}$$

The comultiplication Δ is defined by

$$\Delta(u^q_{ij}) = \sum_k u^q_{ik} \otimes u^q_{kj}.$$

Explicitly, we can write this comultiplication as

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,\ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

2 Kac-Paljutkin Hopf algebra

The Kac–Paljutkin Hopf algebra was introduced by Kac and Paljutkin as the smallest example of semisimple Hopf algebra which is neither commutative (function algebra of finite group) nor cocommutative (group algebra of finite group) [2].

Definition. Kac–Paljutkin algebra $(C(G_{\mathrm{KP}}), \Delta)$ is the eight dimensional Hopf algebra given by

$$C(G_{\mathrm{KP}}) = \mathbb{C}\cdot\epsilon \oplus \mathbb{C}\cdot\alpha \oplus \mathbb{C}\cdot\beta \oplus \mathbb{C}\cdot\gamma \oplus M_2(\mathbb{C}),$$

as an ($*$-)algebra. The comultiplication $\Delta\colon C(G_{\mathrm{KP}}) \to C(G_{\mathrm{KP}}) \otimes C(G_{\mathrm{KP}})$ is defined by

$$\begin{aligned}
\Delta(\epsilon) &= \epsilon\otimes\epsilon + \alpha\otimes\alpha + \beta\otimes\beta + \gamma\otimes\gamma + \frac{1}{2}\sum_{1\le i,j\le 2} \epsilon_{ij}\otimes\epsilon_{ij},\\
\Delta(\alpha) &= \epsilon\otimes\alpha + \alpha\otimes\epsilon + \beta\otimes\gamma + \gamma\otimes\beta\\
&\quad + \frac{1}{2}(\epsilon_{11}\otimes\epsilon_{22} + i\epsilon_{12}\otimes\epsilon_{21} - i\epsilon_{21}\otimes\epsilon_{12} + \epsilon_{22}\otimes\epsilon_{11}),\\
\Delta(\beta) &= \epsilon\otimes\beta + \beta\otimes\epsilon + \alpha\otimes\gamma + \gamma\otimes\alpha\\
&\quad + \frac{1}{2}(\epsilon_{11}\otimes\epsilon_{22} - i\epsilon_{12}\otimes\epsilon_{21} + i\epsilon_{21}\otimes\epsilon_{12} + \epsilon_{22}\otimes\epsilon_{11}),\\
\Delta(\gamma) &= \epsilon\otimes\gamma + \gamma\otimes\epsilon + \alpha\otimes\beta + \beta\otimes\alpha\\
&\quad + \frac{1}{2}(\epsilon_{11}\otimes\epsilon_{11} - \epsilon_{12}\otimes\epsilon_{12} - \epsilon_{21}\otimes\epsilon_{21} + \epsilon_{22}\otimes\epsilon_{22}),\\
\Delta(x) &= \epsilon\otimes x + \alpha\otimes u_\alpha x u_\alpha^* + \beta\otimes u_\beta x u_\beta^* + \gamma\otimes u_\gamma x u_\gamma^*\\
&\quad + x\otimes\epsilon + \overline{u}_\alpha x \overline{u}_\alpha^*\otimes\alpha + \overline{u}_\beta x \overline{u}_\beta^*\otimes\beta + \overline{u}_\gamma x \overline{u}_\gamma^*\otimes\gamma,
\end{aligned}$$

for projections $\epsilon, \alpha, \beta, \gamma$ and $x \in M_2(\mathbb{C})$, where ϵ_{ij} are the matrix units in $M_2(\mathbb{C})$ and

$$u_\alpha = \begin{pmatrix} 0 & i\\ 1 & 0\end{pmatrix},\ u_\beta = \begin{pmatrix} 0 & 1\\ i & 0\end{pmatrix},\ u_\gamma = \begin{pmatrix} -1 & 0\\ 0 & 1\end{pmatrix}.$$

Theorem. [3] There exists a surjective Hopf $*$-homomorphism from $C(SU_{-1}(2))$ to $C(G_{\mathrm{KP}})$.

By the above theorem, the corresponding quantum group G_{KP} can be regarded as a quantum subgroup of $SU_{-1}(2)$, We use the graded twist method of Bichon–Neshveyev–Yamashita [1] as a crucial technique to obtain Hopf $*$-homomorphism from $C(SU_{-1}(2))$. Besides, we employ a key fact that the corepresentation category of the Kac–Paljutkin algebra can be realized as a Tambara–Yamagami tensor category [4] associated with the Krein 4-group, $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

References

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