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Representation theory of compact quantum groups based on operator algebras and its application

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1 Introduction

The theory of operator algebras emerged from quantum mechanics in the study of Murray and von Neumann to give it a mathematical framework. In this theory, the Gelfand–Naimark theorem states that a commutative operator algebra (unital C*-algebra) is isomorphic to the algebra of continuous functions on some compact space. According to this fundamental result, a general operator algebra can be seen as an algebra of continuous functions on a hypothetical "noncommutative" compact topological space.

The theorem of Potryagin states that the dual of a locally compact abelian group G is again a locally compact abelian group $\hat{G}$, which is the group of characters. One of the major motivations of the study in topological quantum groups and Hopf algebras is the extension of Potryagin duality to non-abelian locally compact groups. Aiming at the generalization of Potryagin duality in the setting of C*-algebras and von Neumann algebras, the theory of Kac algebras was introduced by Kac and Vaunermann [17], and by Enock and Schwartz [6]. The C*-algebraic research was conducted by Vallin and Enock [7, 18]. They gave the first extension of Pontryagin duality to all locally compact groups in a framework of Kac algebras in von Neumann setting. These works turned out to be unsatisfactory as their assumption on the antipode was too strong and excluded many interesting examples.

Aiming at overcoming such limitation, the theory of compact quantum groups developed by Woronowicz [20, 21, 23], which gave examples which do not sit in the framework of Kac algebra. The concept of Woronowicz's compact quantum groups was very attractive in the viewpoint of the conciseness of the definition and the striking similarity between its corepresentation theory and the representation theory of compact groups.

In the theory of compact quantum groups, the most important example is the quantum $SU_q(2)$, constructed in [21]. This gives a one-parameter deformation of the algebra of functions on the compact group $SU(2)$ as a C*-algebra with coproduct, which represents a group law on the noncommutative space $SU_q(2)$. The case of $q = 1$ corresponds to the algebra of $SU(2)$.

Parallel to Woronowicz's work, Drinfeld and Jimbo defined q-deformation of semisimple Lie groups, which are new Hopf algebras, by deforming universal enveloping algebras of semisimple Lie algebras through the algebraic study of quantum integrable systems [5, 8]. The C*-algebra of $SU_q(2)$ contains a dense Hopf $*$-algebra of matrix coefficients of unitary corepresentations, which can be regarded as the Hopf dual of the Drinfeld–Jimbo q-deformation Hopf algebra. The theory of semisimple Lie algebras also incorporates with an extension to the case of affine Lie algebras, which can be tackled simultaneously with the usual semisimple algebras in the framework of Kac–Moody Lie algebras. Drinfeld and Jimbo also developed in the case of quantum affine enveloping algebras and, more generally, the quantum analogs of the Kac–Moody Lie algebras [11].

A significant feature of Woronowicz's construction is that the negative range $q < 0$ is allowed, which is different from naively setting q to be a negative number in the Drinfeld–Jimbo construction. In particular, a concrete description for $q = -1$ is given by Zakrzewski's realization of $C(SU_{-1}(2))$ as a C*-subalgebra of $M_2(C(SU(2)))$ [24]. This technique is also useful for computation in K-theory of the algebra $C(SU_{-1}(2))$ [1].

Regarding the quantum subgroups of $SU_{-1}(2)$, a pioneering study was carried out by Podleś [14], who investigated the subgroups and the quotient spaces of quantum $SU(2)$. The complete classification of quantum homogeneous spaces over $SU_q(2)$ realized as coideals are obtained by Tomatsu [16], inspired by Wassermann's classification of ergodic actions of $SU(2)$ [19]. Their results gave a classification of coideals in terms of graphs. According to the McKay correspondence, the homogeneous spaces for $SU(2)$ were classified by the extended Dynkin diagrams, and their results provided a very similar picture.

By Woronowicz's Tannaka–Krein duality [22], a compact quantum group G can be recovered from their representation category $\mathrm{Rep}\, G$, the C*-tensor category of finite dimensional unitary representations, and the fiber functor. The $SU_q(2)$ is distinguished in this respect, because its representation category has a universality for the fundamental representation and the associated morphism which solves its conjugate equations. The Tannaka–Krein duality for compact quantum homogeneous spaces over a compact quantum group G, established by De Commer and Yamashita, says that such homogeneous spaces correspond to module C*-categories over $\mathrm{Rep}\, G$. Such module categories can be also described in terms of tensor functors from $\mathrm{Rep}\, G$ to a category of bi-graded Hilbert spaces [3], the quantum $SU(2)$ case being explained in detail in [4]. The universality of the representation category of $SU_q(2)$ implies that the quantum homogeneous spaces over $SU_q(2)$ are classified by graphs generalizing the McKay correspondence.

The Kac–Paljutkin Hopf algebra was introduced by Kac and Paljutkin as the smallest example of semisimple Hopf algebra which is neither commutative (function algebra of finite group) nor cocommutative (group algebra of finite group) [9]. In this paper, we show that this algebra appears as a quotient of $C(SU_{-1}(2))$. Conceptually, the corresponding quantum group G_{KP} can be regarded as a quantum subgroup of $SU_{-1}(2)$, and the quotient map of Hopf algebras is "restriction" of functions.

A key fact for us is the corepresentation category of the Kac–Paljutkin algebra can be realized as a Tambara–Yamagami tensor category [15] associated with the Krein 4-group, $K_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

We use the graded twist method of Bichon–Neshveyev–Yamashita [2] as another crucial technique to obtain Hopf $*$-homomorphism from $C(SU_{-1}(2))$. This twisting gives a useful description of the Hopf algebra $C(SU_{-1}(2))$ as a deformation of the Hopf algebra $C(SU(2))$, suited to study of its Hopf quotients. We apply their method for describing quantum subgroups of a compact quantum group obtained as the graded twisting of a genuine compact group.

This paper has nine sections including this introduction organized as follows. In Section 2, we give the operator algebraic approach to quantum groups such as the definitions of compact quantum groups and their representation theory. By Woronowicz's definition of a compact quantum group, it becomes possible to generalize the whole theory of compact groups to the quantum group setting. The essential fact is the existence of a unique state on A, that is left and right invariant. The state is called the Haar state of the compact quantum group.

In Section 3, we explain the theory of Hopf $*$-algebras because the theory of compact quantum groups involves the aspects of a purely algebraic nature on the one hand and of a topological nature on the other. For a compact quantum group G, we construct the Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$ of the matrix coefficients of all finite dimensional representations of G. Furthermore, we give a characterization of the Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$.

In Section 4, we introduce C*-tensor categories to understand compact quantum groups from the perspective of categories. The crucial fact is the extension of the Tannaka–Krein duality to the quantum setting. Moreover, we give the definition of module C*-categories.

Section 5 is a preliminary section on the Tambara–Yamagami tensor categories and the graded twisting of Hopf algebras. We also recall a presentation of the Kac–Paljutkin algebra following [15]. we describe the construction of the graded twisting of Hopf algebras and then recall that the Hopf algebra $C(SU_{-1}(2))$ is isomorphic to the graded twisting of $C(SU(2))$.

In Section 6, we give a realization of Kac–Paljutkin Hopf algebra as a quotient of $C(SU_{-1}(2))$. An essential ingredient in our computation is comparison of the two different kinds of projective representations of the Krein

4-group.

In Section 7, we give an explanation of the one-dimensional representation of Kac–Paljutkin quantum group. In addition, we also describe the associated coideal which is one of the type D_4^* discussed in [16], which is also suggested in [14].

Section 8 treats the topics of asymmetric simple exclusion processes (ASEP). We focus on the transition matrices for a usual ASEP and related facts on Temperley–Lieb algebra. Then, we explain the extension to three-state ASEP using the tensor product of the fundamental representation of $SU_q(2)$. The content of this section is based on the talk in the workshop on "Non-commutative Probability and Related Fields" held on 21 November 2019 at Ochanomizu University.

This paper includes the parts of contents in [10], together with background materials and expositions on related works.

2 Compact quantum groups

2.1 Definitions

For general theory of compact quantum groups, we refer to [13]. When we deal with C*-algebras, the symbol $\otimes$ denotes the minimal tensor product.

Definition 1. A compact quantum group is a pair (A, Δ) of an unital C*-algebra A and an unital $*$-homomorphism $\Delta\colon A \to A \otimes A$ called comultiplication such that

1. (coassociativity) $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,
2. (cancelation property) the spaces

$$(A \otimes 1)\Delta(A) = \mathrm{span}\{(a \otimes 1)\Delta(b) | a, b \in A\},$$
$$(1 \otimes A)\Delta(A) = \mathrm{span}\{(1 \otimes a)\Delta(b) | a, b \in A\}$$

are dense in $A \otimes A$.

Example 2. Let G be a compact group. Then a compact quantum group (A, Δ) can be constructed as follows. An unital C*-algebra A is the algebra $C(G)$ of complex valued continuous functions on G. In this case $A \otimes A$ can be identified with $C(G \times G)$ so the comultiplication $\Delta\colon C(G) \to C(G \times G)$ is given by

$$\Delta(f)(g, h) = f(gh) \text{ for all } f \in C(G),\ g, h \in G.$$

Any compact quantum group (A, Δ) with abelian A is of this form. Therefore we write $(C(G), \Delta)$ as a suggestive notation for any compact quantum group, and use G to denote the object behind it.

Definition 3. Let q be a real number such that $|q| \leq 1$, and $q \neq 0$. The quantum $SU(2)$ group $SU_q(2)$ is defined as follows. The algebra $C(SU_q(2))$

is the universal C*-algebra generated by two elements α and γ such that

$$(u^q_{ij})_{i,j} = \begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix} \text{ is unitary.} \tag{1}$$

The comultiplication Δ is defined by

$$\Delta(u^q_{ij}) = \sum_k u^q_{ik} \otimes u^q_{kj}.$$

Explicitly, we can write this comultiplication as

$$\Delta(\alpha) = \alpha \otimes \alpha - q\gamma^* \otimes \gamma,\ \Delta(\gamma) = \gamma \otimes \alpha + \alpha^* \otimes \gamma.$$

From now unless we want to be specific about q we write u_{ij} for u^q_{ij}.

If $q = 1$ then we can get the usual compact group $SU(2)$. For $q \neq 1$ the compact quantum group $SU_q(2)$ can be considered as a deformations of $SU(2)$. We are particularly interested in the case of $q = -1$, in which case the relation (1) becomes

$$\alpha^*\alpha + \gamma^*\gamma = 1,\ \alpha\alpha^* + \gamma\gamma^* = 1,\ \gamma^*\gamma = \gamma\gamma^*,\ \alpha\gamma = -\gamma\alpha,\ \alpha\gamma^* = -\gamma^*\alpha.$$

Definition 4. Let $F \in GL_n(\mathbb{C})$, $n \geq 2$, such that $F\overline{F} = \pm I_n$, where $\overline{F}$ denotes the matrix with entry-wise complex conjugates of F. The algebra $C(O^+_F)$ is the universal C*-algebra generated by $u_{ij}, 1 \leq i, j \leq n$, such that

$$U = (u_{ij})_{i,j} \text{ is unitary and } U = FU^cF^{-1} \quad (U^c = (u^*_{ij})_{i,j}).$$

The comultiplication $\Delta\colon C(O^+_F) \to C(O^+_F) \otimes C(O^+_F)$ is defined by

$$\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}.$$

This compact quantum group O^+_F is called the free orthogonal quantum group associated with F.

We note that $SU_q(2)$ is an example of the free orthogonal quantum group by taking the matrix

$$F = \begin{pmatrix} 0 & -\mathrm{sgn}(q)|q|^{\frac{1}{2}} \\ |q|^{-\frac{1}{2}} & 0 \end{pmatrix}.$$

2.2 Haar state

Let G be a compact quantum group. For bounded linear functionals ω_1, ω_2 on $C(G)$, define the convolution by

$$\omega_1 * \omega_2 = (\omega_1 \otimes \omega_2)\Delta.$$

If G is a genuine group, then this is same as the definition of the convolution of measures on G. In this case, the Haar measure ν on G is characterized by the identity $\mu * \nu = \nu * \mu = \mu(G)\nu$ for any complex measure μ on G.

Theorem 5. For any compact quantum group G, there exists a unique state h on $C(G)$ such that

$$\omega * h = h * \omega = \omega(1)h$$

for any $\omega \in C(G)^*$. The state h is called the Haar state.

Proof. We show the existence of h in several steps.

Step 1. If ω is a state on $C(G)$, then there exists a state h on $C(G)$ such that $\omega * h = h * \omega = h$.

We construct h by taking any weak * limit of the states

$$\frac{1}{n}\sum_{k=1}^{n} \omega^{*k}.$$

Step 2. Let ν be a state on $C(G)$ such that $0 \le \nu \le \omega$, and $\omega * h = h * \omega = \omega(1)h$. Then $\nu * h = h * \nu = \nu(1)h$.

We may assume that $\omega(1) = 1$. Fix an element $a \in C(G)$ and take $b = (\iota \otimes h)\Delta(a)$. Then we have

$$\begin{aligned}&(h \otimes \omega)((\Delta(b) - b \otimes 1)^*(\Delta(b) - b \otimes 1))\\ &= (h * \omega)(b^*b) - (h * \omega)(\Delta(b)^*(b \otimes 1)) - (h * \omega)((b^* \otimes 1)\Delta(b) + h(b^*b)) = 0\end{aligned}$$

By the coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ of the coproduct Δ,

$$\begin{aligned}(\iota \otimes \omega)\Delta(b) &= (\iota \otimes \omega * h)\Delta(a) \\ &= (\iota \otimes h)\Delta(a) \\ &= b.\end{aligned}$$

Therefore, $(h \otimes \nu)((\Delta(b) - b \otimes 1)^*(\Delta(b) - b \otimes 1)) = 0$. By Cauchy-Schwarz inequality, we have

$$(h \otimes \nu)((c \otimes 1)(\Delta(b) - b \otimes 1)) = 0$$

for all C in $C(G)$. Hence we get

$$\begin{aligned}(h \otimes \nu * h)((c \otimes 1)\Delta(a)) &= (h \otimes \nu)((c \otimes 1)\Delta(b)) \\ &= (h \otimes \nu)(cb \otimes 1) \\ &= h(cb)\nu(1) \\ &= \nu(1)(h \otimes h)((c \otimes 1)\Delta(a)). \qquad (2)\end{aligned}$$

Since the space $(C(G) \otimes 1)\Delta(C(G))$ is dense in $C(G) \otimes C(G)$, the equation (2) means that $\nu * h = \nu(1)h$. Similarly, we get $h * \nu = \nu(1)h$.

End of proof. For a finite set $F = \{\omega_1, \ldots, \omega_n\}$ of states on $C(G)$, take

$$\omega_F = \frac{1}{n}(\omega_1 + \cdots + \omega_n).$$

By step 1 and step 2, we can find a state h_F such that $\omega_F * h_F = h_F * \omega_F = h_F$. Hence for every $i = 1, \ldots, n$, we have $\omega_i * h_F = h_F * \omega_i = h_F$. Define h by taking the weak $*$ limit point of the states h_F as F increases. Then this state h is the Haar state. □

2.3 Representations

Definition 6. A representation of a compact quantum group G on a finite dimensional vector space H_U is an invertible element U of $B(H) \otimes C(G)$ such that

$$(\iota \otimes \Delta)(U) = U_{12}U_{13} \text{ in } B(H_U) \otimes C(G) \otimes C(G).$$

The representation U is called unitary if H_U is a Hilbert space and U is unitary. The unitaries $U = (u_{ij})_{i,j}$ in Definition 3 and Definition 4 define unitary representations of each compact quantum group. They are called the fundamental representations of the corresponding quantum groups. We can obviously take direct sums of representations. The tensor product of two finite dimensional representations U and V is the representation $U \times V$ on $H_U \otimes H_V$ defined by $U \times V = U_{13}V_{23}$.

Definition 7. Assume (U, H_U) and (V, H_V) are finite dimensional representations of a compact quantum group G. Then an operator $T\colon H_U \to H_V$ is an intertwiner from U to V if

$$(T \otimes 1)U = V(T \otimes 1).$$

The space of intertwiners from U to V is denoted by $\mathrm{Mor}(U, V)$. A representation (U, H_U) is irreducible if $\mathrm{Mor}(U, U) = \mathbb{C}$.

Proposition 8 (Schur's lemma)**.** Let G be a compact quantum group. Two irreducible unitary representations U and V are either unitarily equivalent and $\mathrm{Mor}(U, V)$ is one-dimensional, or $\mathrm{Mor}(U, V) = 0$.

Proof. Take a nonzero intertwiner $T\colon H_U \to H_V$ so that the scalar T^*T in $\mathrm{End}(U)$ and TT^* in $\mathrm{End}(V)$ are nonzero. Then the operator T is unitary up to a scalar factor. If we take any other intertwiner $S\colon H_U \to H_V$, then T^*S belongs to $End(U)$ and it is a scalar operator. Hence we have $S = \lambda T$ for some $\lambda \in \mathbb{C}$. Therefore, $\mathrm{Mor}(U, V) = \mathbb{C}T$. □

Proposition 9. Every finite dimensional representation of a compact quantum group is equivalent to a unitary representation.

Proof. For a finite dimensional representation $U \in B(H_U) \otimes C(G)$, we take any Hilbert space structure on H_U and take an element $Q = (\iota \otimes h)(U^*U)$ in $B(H_U)$. Since U is invertible, it satisfies that $U^*U \geq \epsilon 1$ for some $\epsilon > 0$. Thus we get $Q \geq \epsilon 1$. Applying $(\iota \otimes h \otimes \iota)$ to the both sides of the equation

$$(\iota \otimes \Delta)(U^*U) = U_{13}^* U_{12}^* U_{12} U_{13}$$

and using $(h \otimes \iota)\Delta(\cdot) = h(\cdot)1$, we obtain

$$Q \otimes 1 = U^*(Q \otimes 1)U.$$

Therefore if we take $V = (Q^{1/2} \otimes 1)U(Q^{-1/2} \otimes 1)$, it is a unitary representation on H_U. We conclude that $Q^{1/2} \in \mathrm{Mor}(U, V)$. □

Theorem 10. Every finite dimensional representation of a compact quantum group G is a direct sum of irreducible representations.

Proof. Let $U \in B(H) \otimes C(G)$ be a finite dimensional representation of G. We may assume that U is unitary. Then the space $\mathrm{End}(U)$ is a C*-algebra. Take a collection of minimal projections $\{e_1, \ldots, e_n\}$ in $\mathrm{End}(U)$ such that $e_1 + \cdots + e_n = 1$. Then $(e_i \otimes 1)U$ are irreducible representations on $e_i H$ and $\bigoplus_i (e_i \otimes 1)U$ is equal to U. □

Definition 11. Let G be a compact quantum group and $U \in B(H) \otimes C(G)$ be a finite dimensional representation of G. Consider the dual space H^* of H, and the map $j\colon B(H) \to B(H^*)$ that sends an operator to the dual operator. Then the contragredient representation to U is the representation U^c on the dual space defined by

$$U^c = (j \otimes \iota)(U^{-1}) \in B(H^*) \otimes C(G).$$

If H is a Hilbert space, then the dual space H^* can be identified with the complex conjugate Hilbert space $\bar{H}$. In that case, the map j is given by $j(T)\bar{\xi} = \overline{T^*\xi}$ for all $T \in B(H)$ and $\bar{\xi} \in \bar{H}$, and it is a $*$-anti homomorphism.

The properties of the contragredient representation U^c to a finite representation U of a compact quantum group G are follows.

1. U^{cc} is equivalent to U.
2. U^{cc} is irreducible if and only if U is irreducible.
3. The flip map $H_U^* \otimes H_V^* \to H_V^* \otimes H_U^*$ defines an equivalence between $(U \times V)^c$ and $V^c \times U^c$.

Definition 12. Let G be a compact quantum group and U be a finite dimensional unitary representation of G. The quantum dimension of U is defined by

$$\dim_q(U) = \mathrm{Tr}(\rho_U).$$

Example 13. Consider a quantum $SU(2)$ group for a real number q such that $|q| \leq 1$ and $q \neq 0$. The fundamental representation $U = (u_{ij})_{i,j}$ for $SU_q(2)$ in the formula (1) is irreducible. The given matrix F for for $SU_q(2)$ satisfies that FUF^{-1} is unitary and $\mathrm{Tr}(F^*F) = \mathrm{Tr}((F^*F)^{-1})$. Thus we get $(F^{-1})^* U^{c*} F^* F U^c F^{-1} = 1$. By the fact that for a finite unitary representation U and an operator $Q \in B(H_U)$, Q is in $\mathrm{Mor}(U, U^{cc})$ if and only if the identity $U^{c*}(j(Q) \otimes 1)U^c = j(Q) \otimes 1$ hold, we can conclude that $j(F^*F) = (F^*F)^t$ belongs to $\mathrm{Mor}(U, U^{cc})$. Hence we get $\rho_U = (F^*F)^t$. More concretely, the matrix F is given by

$$F = \begin{pmatrix} 0 & -\mathrm{sgn}(q)|q|^{\frac{1}{2}} \\ |q|^{-\frac{1}{2}} & 0 \end{pmatrix}$$

Thus we obtain the operator ρ_U as

$$\rho_U = (F^*F)^t = \begin{pmatrix} |q|^{-1} & 0 \\ 0 & |q| \end{pmatrix}.$$

Therefore, the quantum dimension of U is $\dim_q U = \mathrm{Tr}(\rho_U) = |q + q^{-1}|$.

Proposition 14 (Orthogonality relations)**.** Let G be a compact quantum group, U be an irreducible finite dimensional representation of G, whose matrix form with respect to an orthonormal basis in H_U is $(u_{ij})_{i,j}$, and let $\rho = \rho_U$. Then

1. The equalities
$$h(u_{kl}u_{ij}^*) = \frac{\delta_{ki}\rho_{jl}}{\dim_q U}, \quad h(u_{ij}^* u_{kl}) = \frac{\delta_{jl}(\rho^{-1})_{ki}}{\dim_q U}.$$
hold.

2. If $V = (v_{kl})_{k,l}$ is an irreducible unitary representation which is not equivalent to U, then $h(v_{kl}u_{ij}^* = h(u_{ij}^* v_{kl})) = 0$.

Proof. (1) For any operator $T \in B(H_U)$, we have that $(\iota \otimes h)(U(T \otimes 1)U^*) \in \operatorname{End}(U) = \mathbb{C}1$. Therefore, there exists a unique positive operator $\rho_r \in B(H_U)$ such that

$$\operatorname{Tr}(\rho_r T)1 = (\iota \otimes h)(U(T \otimes 1)U^*)$$

for all $T \in B(H_U)$. Define the operators Q_r and Q_l in $B(\bar{H}_U)$ defined by

$$Q_r = (\iota \otimes h)(U^{c*}U^c), \quad Q_l = (\iota \otimes h)(U^c U^{c*}).$$

By the property of taking the trace, $\operatorname{Tr}(XYZ) = \operatorname{Tr}(YZX) = \operatorname{Tr}(j(Y)j(Z)j(X))$, we get that

$$\begin{aligned}
(\dim_q U)\operatorname{Tr}(\rho_r T) &= (\operatorname{Tr} \otimes h)(U(T \otimes 1)U^*) \\
&= (\operatorname{Tr} \otimes h)((j \otimes \iota)(U)(j \otimes \iota)(U^*)(j(T) \otimes 1)) \\
&= (\operatorname{Tr} \otimes h)(U^{c*}U^c(j(T) \otimes 1)) \\
&= \operatorname{Tr}(Q_r j(T)) \\
&= \operatorname{Tr}(j(Q_r)T).
\end{aligned}$$

Since $Q_r = \frac{\dim U}{\dim_q U} j(\rho_U)$, we have

$$\rho_U = \frac{j(Q_r)}{\dim U} = \frac{\rho}{\dim_q U}.$$

Applying $T = m_{ij}$ to the identity $\mathrm{Tr}(\rho_r T)T = (\iota \otimes h)(U(T \otimes 1)U^*)$, we obtain the first equality. Similarly, showing that

$$\frac{\mathrm{Tr}(\rho^{-1}T)}{\dim_q U} = 1 = (\iota \otimes h)(U^*(T \otimes 1)U)$$

for all $T \in B(H_U)$ leads the second equality.

(2) Let V be an irreducible unitary representation such that $\mathrm{Mor}(V, U) = 0$ and $\mathrm{Mor}(U, V) = 0$. Then we get that

$$(\iota \otimes h)(V(S \otimes 1)U^*) = 0, \quad (\iota \otimes h)(U^*(T \otimes 1)V) = 0$$

for all $S\colon H_U \to H_V$ and $T\colon H_V \to H_U$. This is equivalent to $h(v_{kl}u_{ij}^*) = 0$ and $h(u_{ij}^* v_{kl}) = 0$. □

Proposition 15. Let G be a compact quantum group and U be a finite dimensional unitary representation. Then there exists a unique positive invertible operator $\rho \in \mathrm{Mor}(U, U^{cc})$ such that

$$\mathrm{Tr}(\cdot \rho) = \mathrm{Tr}(\cdot \rho^{-1}) \quad \text{on } \mathrm{End}(U) \subset B(H_U).$$

We denote by $\rho_U \in B(H_U)$ this operator. By definition, $\dim_q U = \mathrm{Tr}(\rho_U)$.

Proof. There exist pairwise nonequivalent irreducible unitary representations U_i, $1 \leq i \leq n$, such that U decomposes into a direct sum of copies of U_i. In other word, $H_U = \bigoplus_i (K_i \otimes H_{U_i})$ for some finite dimensional Hilbert space K_i, and $U = \bigoplus_i (1 \otimes U_i)$. Then we have $\mathrm{End}(U) = \bigoplus_i (B(K_i) \otimes 1)$. Therefore, the operator $\rho = \bigoplus_i (1 \otimes \rho_{U_i})$ is positive and invertible, and it satisfies $\mathrm{Tr}(\cdot \rho) = \mathrm{Tr}(\cdot \rho^{-1})$.

Assume $\rho' \in B(H_U)$ is another positive invertible operator with the above property. Then $\rho^{-1}\rho' \in \mathrm{End}(U)$, and we have $\rho' = \bigoplus_i (T_i \otimes \rho_{U_i})$ for some positive invertible operator $T \in B(K_i)$. Therefore $\mathrm{Tr}(\cdot T_i) = \mathrm{Tr}(\cdot T_i^{-1})$ on $B(K_i)$, so we get $T_i = T_i^{-1}$. Hence $T_i = 1$ because T_i is positive. Thus we get $\rho = \rho'$. □

Definition 16. Let G be a compact quantum group, U be a finite dimensional unitary representation of G on H_U, and $\rho_U \in B(H)$ be a normalized positive invertible operator such that $\mathrm{Mor}(U, U^{cc})$ is spanned by ρ_U. Then the conjugate representation to U is defined by

$$\bar{U} = (j(\rho_U)^{1/2} \otimes 1)U^c(j(\rho_U)^{-1/2} \otimes 1)$$

in $B(\bar{H}_U) \otimes C(G)$. This is the canonical unitary form of the contragredient representation U^c. The morphism ρ_U is equal to 1 if and only if the contragredient representation U^c is unitary.

Let us give a list of the properties of the conjugate representation. For any finite dimensional unitary representation U of a compact quantum group G, we have the following properties.

1. $\bar{\bar{U}} = U$
2. $\overline{U \oplus V} = \bar{U} \oplus \bar{V}$
3. the flip map $\bar{H}_U \otimes \bar{H}_V \to \bar{H}_U \otimes \bar{H}_V$ defines an equivalence between $\overline{U \times V}$ and $\bar{V} \times \bar{U}$.

Proposition 17. Let G be a compact quantum group. Then for any finite dimensional unuitary representation U of G, we have that $\rho_{\bar{U}} = j(\rho_U)^{-1}$. In particular, $\dim_q \bar{U} = \dim_q U$.

Proof. For an irreducible representation U, we can compute that

$$\begin{aligned}
(\bar{U})^c &= (j \otimes \iota)(\bar{U})^* \\
&= (j \otimes \iota)(j(\rho_U)^{-1/2} \otimes 1)U^{c*}(j(\rho_U)^{-1/2} \otimes 1)) \\
&= (\rho_U^{1/2} \otimes 1)(j \otimes \iota)(U^{c*})(\rho_U^{-1/2} \otimes 1) \\
&= (\rho_U^{1/2} \otimes 1)U(\rho_U^{-1/2} \otimes 1).
\end{aligned}$$

Therefore, $(\rho_U^{1/2} \otimes 1)(\bar{U})^c(\rho_U^{1/2} \otimes 1)$ is unitary. It follows that $j(\rho_U)^{-1}$ is in $\mathrm{Mor}(\bar{U}, \bar{U}^{cc})$. Using an equality $\mathrm{Tr}(\rho_U) = \mathrm{Tr}(\rho_U^{-1})$, we get $\rho_{\bar{U}} = j(\rho_U)^{-1}$. $\square$

3 Hopf $*$-algebras

3.1 Definitions and examples

Definition 18. A pair (A, Δ) consisting of a unital $*$-algebra A and a unital $*$-homomorphism $\Delta\colon A \to A \otimes A$ is called a Hopf $*$-algebra if Δ satisfies the coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$ and there exist linear maps $\epsilon\colon A \to \mathbb{C}$ and $S\colon A \to A$ such that

$$(\epsilon \otimes \iota)\Delta(a) = (\iota \otimes \epsilon)\Delta(a) = a \text{ and } m(S \otimes \iota)\Delta(a) = m(\iota \otimes S)\Delta(a) = \epsilon(a)1$$

for all $a \in A$, where $m\colon A \otimes A \to A$ is the multiplication map. The map ϵ is called a counit and S is called an antipode.

Let us list a number of properties of the maps ϵ and S that follow from the axioms:

1. ϵ and S are uniquely determined;
2. $\epsilon S = \epsilon$;
3. ϵ is a $*$-homomorphism and S is an anti-homomorphism;
4. $\Delta S = (S \otimes S)\Delta^{op}$;
5. $S(S(a^*)^*) = a$ for all $a \in A$.

Example 19. Let G be a compact quantum group. For a unitary representation U, the element $(\omega_{\xi,\zeta} \otimes \iota)(U) \in C(G)$ for $\xi, \zeta \in H_U$ are called the matrix coefficients of U, where $\omega_{\xi,\zeta}\colon B(H_U) \to \mathbb{C}$ is a linear functional defined by $\omega_{\xi,\zeta}(T) = (T\xi, \zeta)$ for $T \in B(H_U)$. It is known that the linear span of matrix coefficients of all finite dimensional representations of G, denoted by $\mathbb{C}[G]$, is a dense $*$-subalgebra of $C(G)$.

The pair $(\mathbb{C}[G], \Delta)$ forms a Hopf $*$-algebra. In order to prove it, define linear maps ϵ, S by

$$(\iota \otimes \epsilon)(U) = 1, \quad (\iota \otimes S)(U) = U^{-1}$$

for every finite dimensional representation U of G. We take representitative U_α of the equivalence classes of irreducible finite dimensional representations. By the orthogonality relations, the matrix coefficients u^α_{ij} with respect to a fixed orthonormal basis in H_{U_α} form a linear basis in $\mathbb{C}[G]$. Therefore, we can define linear maps ϵ, S by

$$\epsilon(u^\alpha_{ij}) = \delta_{ij}, \quad S(u^\alpha_{ij}) = u^{\alpha *}_{ji}.$$

For any finite dimensional finite dimensional representation U, it decomposes into a direct sum of copies of U_α. Hence, it satisfies

$$(\iota \otimes \epsilon)(U) = 1, \quad (\iota \otimes S)(U) = U^{-1}.$$

Applying $(\iota \otimes \epsilon \otimes \iota)$ to the equation $(\iota \otimes \Delta)(U) = U_{12}U_{13}$, we get

$$(\iota \otimes \epsilon \otimes \iota)(\iota \otimes \Delta)(U) = (\iota \otimes \epsilon \otimes \iota)(U_{12}U_{13}) = U.$$

It leads $(\epsilon \otimes \iota)\Delta = \iota$. Similarly, we compute

$$(\iota \otimes \iota \otimes \epsilon)(\iota \otimes \Delta)(U) = (\iota \otimes \iota \otimes \epsilon)(U_{12}U_{13}) = U$$

and we get $(\iota \otimes \epsilon)\Delta = \iota$. On the other hand, by applying $(\iota \otimes m)(\iota \otimes S \otimes \iota)$ to the same equation, we have

$$\begin{aligned}(\iota \otimes m)(\iota \otimes S \otimes \iota)(\iota \otimes \Delta)(U) &= (\iota \otimes m)(U^{-1}_{12}U_{13}) \\ &= U^{-1}U = 1 = (\iota \otimes \epsilon(\cdot)1)(U).\end{aligned}$$

Therefore, we get $m(S \otimes \iota)\Delta = \epsilon(\cdot)1$. Similarly, $m(\iota \otimes S)\Delta = \epsilon(\cdot)1$ can be checked. Thus $(\mathbb{C}[G], \Delta)$ is a Hopf *-algebra.

Definition 20. Let (A, Δ) be a Hopf $*$-algebra and H be a vector space. A corepresentation of A on H is a linear map $\delta\colon H \to H \otimes A$ such that

$$(\delta \otimes \iota)\delta = (\iota \otimes \delta)\delta, \quad (\iota \otimes \epsilon)\delta = \iota.$$

The corepresentation δ is called unitary if H is a Hilbert space and the equation $\langle \delta(\xi), \delta(\zeta) \rangle = (\xi, \zeta)1$ holds for all $\xi, \zeta \in H$, where $\langle \cdot, \cdot \rangle$ is defined by

$$\langle \xi \otimes a,\ \zeta \otimes b \rangle = (\xi, \zeta) b^* a$$

for all $a, b \in A$.

A subspace H' of H is said to be invariant if $\delta(H')$ is contained in $H' \otimes A$. The corepresentation is called irreducible if there are no proper invariant subspace.

If $\delta\colon H \to H \otimes A$ is a corepresentation on a finite dimensional space, then $\delta(\xi) = U(\xi \otimes 1)$ for a uniquely determined element $U \in B(H) \otimes A$. Namely, if $\{\xi_i\}_{i=1}^n$ is a basis in H and $\delta(\xi_i) = \sum_i \xi \otimes u_{ij}$, then $U = \sum_{i,j} m_{ij} \otimes u_{ij}$. Conversely, any element $U \in B(H) \otimes A$ with properties

$$(\iota \otimes \Delta)(U) = U_{12} U_{13}, \quad (\iota \otimes \epsilon)(U) = 1$$

defines a corepresentation.

All results on finite dimensional representations of compact quantum groups can be extended to finite dimensional corepresentations of Hopf $*$-algebras. Unitarity of δ is the same as unitarity of U. In particular, a finite dimensional unitary corepresentation of $(\mathbb{C}[G], \Delta)$ is the same thing as a finite dimensional unitary representation of G. The irreducibility of δ is equivalent to the irreducibility of U.

3.2 Characterization of Hopf *-algebras arising from compact quantum groups

We will give a characterization of Hopf *-algebras that arises from compact quantum groups.

Lemma 21. Let A be a Hopf *-algebra. For any finite or infinite dimensional corepresentaion $\delta\colon H \to H \otimes A$, we have

$$H \otimes A = \delta(H)(1 \otimes A).$$

Proof. Define linear maps $S\colon H \otimes A \to H \otimes A$ and $r\colon H \otimes A \to H \otimes A$ by

$$S(\xi \otimes a) = (\iota \otimes S)\delta(\xi)(1 \otimes a), \quad r(\xi \otimes a) = \delta(\xi)(1 \otimes a).$$

We claim that $rs = \iota$. Since the maps r, S are right A-module maps, it suffices to compute rs on $\xi \otimes 1$. Then we get that

$$\begin{aligned} rs(\xi \otimes 1) &= r((\iota \otimes S)\delta(\xi)) \\ &= (\iota \otimes m(\iota \otimes S))(\delta \otimes \iota)\delta(\xi) \\ &= (\iota \otimes m(\iota \otimes S))(\iota \otimes \Delta)\delta(\xi) \\ &= (\iota \otimes \epsilon(\cdot)1)\delta(\xi) \\ &= \xi \otimes 1. \end{aligned}$$

It shows that $rs = \iota$. Thus the map r is surjective. We can also check that the map r is injective and thus it concludes the proof. □

In particular, if the corepresentation δ is unitary and K is a closed invariant subspace of H, then the space $K^{\perp}$ is also invariant. Indeed, for elements $\xi \in K^{\perp}$ and $a \in A$, we have

$$\langle \delta(\xi), \delta(\zeta)(1 \otimes a) \rangle = (\xi, \zeta)a^* = 0,$$

and then we get $\langle \delta(\xi), \zeta \otimes 1 \rangle = 0$ for any ζ in K since $\delta(K)(1 \otimes A) = K \otimes A$. Therefore, $\delta(\xi)$ is in $K^{\perp} \otimes A$.

Thus, any finite dimensional corepresentation decomposes into a direct sum of finite dimensional unitary corepresentations.

Consider a dual space $\mathcal{U} = A^*$. It is a unital *-algebra with the product given by $\omega\nu = (\omega \otimes \nu)\Delta$, the involution given by $\omega^* = \bar{\omega}S$, and the unit ϵ.

For every finite dimensional corepresentation $U \in B(H) \otimes A$, define a unital representation $\pi_U : \mathcal{U} \to B(H)$ by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. A subspace K of H is invariant if and only if K is $\pi_U(\mathcal{U})$-invariant. When the corepresentation U is unitary, then π_U is a $*$-representation and $\pi_U(\omega)^* = (\iota \otimes \omega)(U^*) = (\iota \otimes \bar{\omega})(U^*)$. Therefore, if the subspace K is $\pi_U(\mathcal{U})$-invariant, then the space $K^\perp$ is also $\pi_U(\mathcal{U})$-invariant.

Theorem 22. Let (A, Δ) be a Hopf *-algebra such that A is generated as an algebra by the matrix coefficients of finite dimensional unitary corepresentations. Then, $(A, \Delta) = (\mathbb{C}[G], \Delta)$ for some compact quantum group G.

Proof. Assume $\mathcal{A}$ is a C*-enveloping algebra of A. This is well-defined since A is generated by matrix coefficients of unitary matircies over A and the matrix coefficients have universal bounds on the norms for all possible *-representations on Hilbert spaces. The most important point in this proof is to show that the canonical homomorphism $A \to \mathcal{A}$ is injective. We will show that there exists a faithful state h on A. It plays a role of the Haar state. In the following, we construct h in several steps.

Step 1. There exists a unique linear functional h such that $h(1) = 1$, $(\iota \otimes h)\Delta(a) = h(a)1$, $(h \otimes 1)\Delta(a) = h(a)1$ for every $a \in A$.

Firstly we show that if $U_1 \in B(H_1) \otimes A, \cdots, U_n \in B(H_n) \otimes A$ are pairwise nonequivalent finite dimensional irreducible corepresentations of A, then the matrix coefficients of $U_1, \ldots, U_n$ with respect to fixed basis in $H_1, \ldots, H_n$ are linearly independent. Define representations $\pi_{U_1}, \ldots, \pi_{U_n}$ of $\mathcal{U} = A^*$ by $\pi_{U_i}(\omega) = (\iota \otimes \omega)(U_i)$. Then they are irreducible and pairwise nonequivalent. Thus, by Jacobson's density theorem, a homomorphism $\bigoplus_i \pi_{U_i} : \mathcal{U} \to \bigoplus_i B(H_i)$ is surjective. By the dimension reason, it happens in only the case where the matrix coefficients of $U_1, \ldots, U_n$ are linearly independent. By the assumption, A is generated by matirx coefficients of finite dimensional unitary corepresentations. A product of matrix coefficients is the matrix coefficients of the tensor product $U \times V = U_{13}V_{23}$ of corepresentations. Since every finite dimensional unitary corepresentation U decomposes into a direct sum of irreducible corepresentations, A is spanned by matrix coefficients of finite dimensional irreducible unitary corepresentations. Take representatives U_α of equivalence classes of finite dimensional irreducible unitary corepresentations of A. Then the matrix coefficients of U_α with respect to any basis in H_{U_α} form a basis in A. Therefore, we define a linear functional h on A such that

$$h(1) = 1, \quad (\iota \otimes h)(U_\alpha) = 0 \text{ if } U_\alpha = 1.$$

By the equation $(\iota \otimes \Delta)(U_\alpha) = (U_\alpha)_{12}(U_\alpha)_{13}$, the functional h has the required properties. The uniqueness of h is obvious.

Step 2. (Orthogonality relations) For every α there exists a positive invertible operator $Q_\alpha \in B(\bar{H}_{U_\alpha})$ such that

$$(\iota \otimes h)(U_\alpha^*(T \otimes 1)U_\beta) = \delta_{\alpha,\beta}\frac{\mathrm{Tr}(Tj(Q_\alpha))}{\dim U_\alpha}1$$

for any $T \in B(H_{U_\beta}, H_{U_\alpha})$.

Define an operator $Q_\alpha = (\iota \otimes h)(U_\alpha^{c*}U_\alpha^c)$ using the contragredient corepresentation $U_\alpha^c = (j \otimes \iota)(U_\alpha^{-1}) = (j \otimes S)(U_\alpha)$ to U_α. The above orthogonality relations holds with this operator Q_α. We show that the operator Q_α is positive invertible for every α. Since we have that $j(Q_\alpha) \in \mathrm{Mor}(U_\alpha, U_\alpha^{cc})$, if U_α is irreducible, then the space $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ is at most one-dimensional and every nonzero operator in $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ is invertible. Because of the fact that $\mathrm{Tr}(j(Q_\alpha)) = \dim U_\alpha > 0$, it is enough to show that the space $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ contains a nonzero positive operator. Since

$$\pi_{U_\alpha^c}(\omega) = j(\pi_{U_\alpha}(\omega S))$$

and the antipode S is bijective, the contragredient corepresentation U_α^c is irreducible. By the fact that the matrix coefficients of U_α are linearly independent, there exists a finite dimensional irreducible unitary corepresentation V among U_β that is equivalent to U_α^c. Choose an invertible operator $T \in \mathrm{Mor}(U_\alpha^c, V)$, then $j(T)$ is in $\mathrm{Mor}(V^c, U_\alpha^{cc})$. On the other hand, by taking adjoint and applying $(j \otimes \iota)$ to the identity $(T \otimes 1)U_\alpha^c = V(T \otimes 1)$, we get

$$\begin{aligned}(j(T)^* \otimes 1)U_\alpha^c &= (j \otimes \iota)(V^*)(j(T)^* \otimes 1)\\ &= V^c(j(T) \otimes 1).\end{aligned}$$

Thus, $j(T)^*$ is in $\mathrm{Mor}(U_\alpha, V^c)$. Hence, $\mathrm{Mor}(U_\alpha, U_\alpha^{cc})$ contains the positive invertible operator $j(T)j(T)^*$.

Step 3. For every nonzero element $a \in A$, we have $h(a^*a) \geq 0$.

Let u_{ij}^α be the matrix coefficients of U_α with respect to an orthonormal basis in which the positive invertible operators $j(Q_\alpha)$ are diagonal. By step 2, u_{ij}^α form orthogonal basis in A with respect to the sesquilinear form $(a, b) = h(b^*a)$. By $(u_{ij}^\alpha, u_{ij}^\alpha) \geq 0$, $(a, b) = h(b^*a)$ is positive definite.

End of proof. Define a left action of A on A by $l(a)(b) = ab$ seeing A as a pre Hilbert space with the scalar product $(a, b) = h(b^*a)$. It is a faithful $*$-representation of A on A. This is a representation by bounded operators. Hence, A is spanned by the matrix coefficients of unitary matrices over A. Since every entry of a unitary matrix must act as an operator of norm less than 1, this representation can be extend to a faithful representation on the Hilbert space completion of A. Therefore, A is considered as a subalgebra of $\mathcal{A}$. The coproduct $\Delta \colon A \otimes A \to A$ can be extended to $\Delta \colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Applying Lemma 21 with $\delta = \Delta$, we get $(1 \otimes \mathcal{A})\Delta(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}$. Similarly, $(\mathcal{A} \otimes 1)\Delta(\mathcal{A}) = \mathcal{A} \otimes \mathcal{A}$. Thus, $(\mathcal{A}, \Delta)$ has the cancellation property. Hence, it is a compact quantum group G. Since A is dense in $\mathbb{C}[G]$ and A is spanned by the matrix coefficients of irreducible unitary representations of G, by the orthogonality relations, we get $A = \mathbb{C}[G]$. □

For every finite dimensional Hopf *-algebra A, its dual space $\mathcal{U} = A^*$ is a $*$-algebra with the product $\omega\nu = (\omega \otimes \nu)\Delta$ and the involution $\omega^* = \bar{\omega}S$. Then

$(\mathcal{U}, \hat{\delta})$ is a Hopf *-algebra with the coproduct, the antipode, and the counit given by $\hat{\delta}(\omega)(A \otimes b) = \omega(ab)$ for $a, b \in A$, $\hat{S}(\omega) = \omega S$, and $\hat{\epsilon}(\omega) = \omega(1)$. It is called the dual of (A, Δ). The dual of $(\mathcal{U}, \hat{\delta})$ is (A, Δ).

Definition 23. The algebra of functions on the dual discrete quantum group $\hat{G}$ of a compact quantum group G is the *-algebra $\mathcal{U}(G) = \mathbb{C}[G]^*$ with the multiplication $\omega\nu = \omega * \nu = (\omega \otimes \nu)\Delta$ and the involution $\omega^* = \bar{\omega}S$.

For every finite dimensional representation U of G, a representation π_U of $\mathcal{U}(G)$ is given by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. If U is unitary, then π_U is $*$-preserving. Fix the representatives U_α of equivalence classes of irreducible unitary representations of G. Then $\mathbb{C}[G]$ is a direct sum of spaces spanned by the matrix coefficients of U_α. Therefore, π_{U_α} define a $*$-homomorphism

$$\mathcal{U}(G) \simeq \prod_\alpha B(H_{U_\alpha}).$$

Let $\mathcal{U}(G^n)$ be the dual space of $\mathbb{C}[G]^{\otimes n}$. Similarly to $\mathcal{U}(G)$ case, $\mathcal{U}(G^n)$ is a $*$-algebra which is canonically isomorphic to

$$\prod_{\alpha_1, \ldots, \alpha_n} B(H_{U_{\alpha_1}} \otimes \cdots \otimes H_{U_{\alpha_n}}).$$

Define $\hat{\Delta}$ by $\hat{\Delta}(\omega)(a \otimes b) = \omega(ab)$ for any $a, b \in \mathbb{C}[G]$. Then $\hat{\Delta}$ is a unital $*$-homomorphism. Equivalently, $\hat{\Delta}(\omega) \in \mathcal{U}(G \times G)$ is a unique element such that $(\pi_{U_\alpha} \otimes \pi_{U_\beta})\hat{\Delta}(\omega)T = T\pi_{U_\gamma}(\omega)$ for any $T \in \mathrm{Mor}(U_\gamma, U_\alpha \times U_\beta)$. In general, $\hat{(}\Delta)(\omega)$ is not in the algebraic tensor product $\mathcal{U}(G) \otimes \mathcal{U}(G) \subset \mathcal{U}(G \times G)$. Define linear maps $\hat{\epsilon}\colon \mathcal{U}(G) \to \mathbb{C}$ and $\hat{S}\colon \mathcal{U}(G) \to \mathcal{U}(G)$ by $\hat{\epsilon}(\omega) = \omega(1)$ and $\hat{S}(\omega) = \omega S$. Then the maps $\hat{\Delta}, \hat{\epsilon}, \hat{S}$, and the multiplication map $m\colon \mathcal{U}(G) \otimes \mathcal{U}(G) \to \mathcal{U}(G)$ can be applied to the factors of $\mathcal{U}(G)^{\otimes n}$. and then they can be extend to $\mathcal{U}(G^n)$. For example, $\iota \otimes \hat{\Delta}\colon \mathcal{U}(G^2) \to \mathcal{U}(G^3)$ by $(\iota \otimes \hat{\Delta})(\omega)(a \otimes b \otimes c) = \omega(a \otimes bc)$. Therefore, $(\mathcal{U}(G), \hat{\Delta})$ is a Hopf $*$-algebra.

4 C*-Tensor categories

4.1 C*-Tensor categories and tensor functors

Definition 24. A category $\mathcal{C}$ is a C*-category if

1. For all objects U, V in $\mathcal{C}$ the space of morphisms $\mathrm{Mor}(U, V)$ from U to V is a Banach space, the map $\mathrm{Mor}(V, W) \times \mathrm{Mor}(U, V) \to \mathrm{Mor}(U, W), (S, T) \mapsto ST$ is bilinear, and $\|ST\| \leq \|S\|\|T\|$;

2. $*\colon \mathcal{C} \to \mathcal{C}$ is antilinear contravariant functor that is the identity map on objects, so if $T \in \mathrm{Mor}(U, V)$ then $T^* \in \mathrm{Mor}(V, U)$, satisfying

   (a) $T^{**} = T$ for any morphisms T.

   (b) $\|T^*T\| = \|T\|^2$ for any $T \in \mathrm{Mor}(U, V)$. In particular, $\mathrm{End}(U) = \mathrm{Mor}(U, U)$ is a unital C*-algebra for any object U.

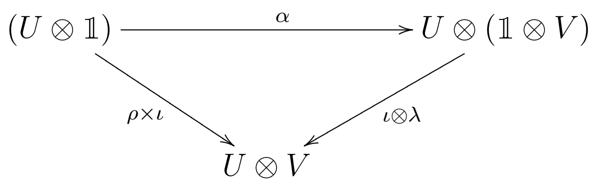
   (c) for any morphism T, the element $T^*T \in \mathrm{End}(U)$ is positive.

The category $\mathcal{C}$ is a C*-tensor category if in addition we are given a bilinear bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}, (U, V) \mapsto U \otimes V$, natural unitary isomorphisms $\alpha_{U,V,W}\colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$, called the associativity morphisms, an object $\mathbb{1}$, called the unit object, and natural unitary isomorphisms $\lambda_U\colon \mathbb{1} \otimes U \to U, \rho_U\colon U \otimes \mathbb{1} \to U$, such that

1. the pentagon diagram commutes.

$$
\begin{array}{ccccc}
 & & ((U \otimes V) \otimes W) \otimes X & & \\
 & \swarrow^{\alpha\otimes\iota} & & \searrow^{\alpha_{12,3,4}} & \\
(U \otimes (V \otimes W)) \otimes X & & & & (U \otimes V) \otimes (W \otimes X) \\
\downarrow{\scriptstyle \alpha_{a,23,4}} & & & & \downarrow{\scriptstyle \alpha_{1,2,34}} \\
U \otimes ((V \otimes W) \otimes X) & & \xrightarrow[\iota\otimes\alpha]{} & & U \otimes (V \otimes (W \times X))
\end{array}
$$

2. $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}}$ and the triangle diagram commutes.



3. $(S \otimes T)^* = S^* \otimes T^*$ for an morphisms S, T.

4. (Direct sum) For any objectsU_1, U_2 there exist an object V and isometries$u_1 \in \mathrm{Mor}(U_1, V)$ and $u_2 \in \mathrm{Mor}(U_2, V)$ such that $u_1 u_1^* + u_2 u_2^* = 1$.

5. (Subobject) For every projection $p \in \mathrm{End}(U)$ there exists an object V and $v \in \mathrm{Mor}(V, U)$ such that $vv^* = p$.

6. $\mathrm{End}(\mathbb{1}) = \mathbb{C}1 \simeq \mathbb{C}$.

7. The category is small, that is, the class of objects is a set.

The category $\mathcal{C}$ is called strict if

$$(U \otimes V) \otimes W = U \otimes (V \otimes W), \quad \mathbb{1} \otimes U = U \otimes \mathbb{1} = U$$

for any objects U, V, W in $\mathcal{C}$ and α, λ, ρ are the identity morphisms. By the result of Mac Lane, any tensor category can be strictified.

Example 25. The category Hilb_f of finite dimensional Hilbert spaces is a strict C*-tensor category with usural tensor product of Hilbert spaces. Morphisms $\mathrm{Mor}(H, K)$ is bounded operators $B(H, K)$ from H to K. The unit object is $\mathbb{C}$.

Example 26. The category $\mathrm{Rep}\, G$ of finite dimensional unitary representations of a compact quantum group G is a strict C*-tensor category. Morphisms $\mathrm{Mor}(H_\pi, H_\rho)$ from π to ρ are intertwiners.

Definition 27. Let $\mathcal{C}$ and $\mathcal{C}'$ be C*-tensor categories. A tensor functor $\mathcal{C} \to \mathcal{C}'$ is a functor $F\colon \mathcal{C} \to \mathcal{C}'$ that is linear on morphisms, together with an isomorphism $F_0\colon \mathbb{1}_{\mathcal{C}'} \to F(\mathbb{1}_{\mathcal{C}})$ in $\mathcal{C}'$ and natural isomorphisms

$$F_2\colon F(U) \otimes F(V) \to F(U \otimes V)$$

such that the following diagrams commute. A tensor functor $F\colon \mathcal{C} \to \mathcal{C}'$ is

$$\begin{array}{ccccc}
(F(U) \otimes F(V)) \otimes F(W) & \xrightarrow{F_2 \otimes \iota} & F(U \otimes V) \otimes F(W) & \xrightarrow{F_2} & F((U \otimes V) \otimes W) \\
\downarrow{\scriptstyle \alpha'} & & & & \downarrow{\scriptstyle F(\alpha)} \\
F(U) \otimes (F(V) \otimes F(W)) & \xrightarrow[\iota \otimes F_2]{} & F(U) \otimes F(V \otimes W) & \xrightarrow[F_2]{} & F(U \otimes (V \otimes W))
\end{array}$$

$$\begin{array}{ccc}
F(\mathbb{1}) \otimes F(U) & \xrightarrow{F_2} & F(\mathbb{1} \otimes U) \\
\uparrow{\scriptstyle F_0 \otimes \iota} & & \downarrow{\scriptstyle F(\lambda)} \\
\mathbb{1}' \otimes F(U) & \xrightarrow{\lambda'} & F(U)
\end{array}
\qquad
\begin{array}{ccc}
F(U) \otimes F(\mathbb{1}) & \xrightarrow{F_2} & F(U \otimes \mathbb{1}) \\
\uparrow{\scriptstyle \iota \otimes F_0} & & \downarrow{\scriptstyle F(\rho)} \\
F(U) \otimes \mathbb{1}' & \xrightarrow{\rho'} & F(U)
\end{array}$$

unitary if in addition $F(T^*) = F(T)^*$ for all morphisms T in $\mathcal{C}$, and F_2 and F_0 are unitary.

Example 28. Let G be a compact quantum group. Then we have a tensor functor $F\colon \operatorname{Rep} G \to \mathrm{Hilb}_f$ defined by $F(U) = H_U$ for every finite dimensional unitary representation U of G. The action of F on morphisms and F_2 are taken to be the identity maps.

Definition 29. A natural isomorphism $\eta\colon F \to G$ between two tensor functors $F, G\colon \mathcal{C} \to \mathcal{C}'$ is said to be monoidal if the diagrams commutes.

$$\begin{array}{ccc}
F(U) \otimes F(V) & \xrightarrow{F_2} & F(U \otimes V) \\
\downarrow{\scriptstyle \eta \otimes \eta} & & \downarrow{\scriptstyle \eta} \\
G(U) \otimes G(V) & \xrightarrow{G_2} & G(U \otimes V)
\end{array}
\qquad
\begin{array}{ccc}
 & \mathbb{1}' & \\
{\scriptstyle F_0} \swarrow & & \searrow {\scriptstyle G_0} \\
F(\mathbb{1}) & \xrightarrow{\eta} & G(\mathbb{1})
\end{array}$$

Definition 30. Two C*-tensor categories $\mathcal{C}$ and $\mathcal{C}'$ are monoidally equivalent if there exist tensor functor $F\colon \mathcal{C} \to \mathcal{C}'$ and $G\colon \mathcal{C}' \to \mathcal{C}$ such that FG and GF are naturally monoidally equivalent if we can take F, G and the natural isomorphisms $FG \simeq \iota$ and $GF \simeq \iota$ to be unitary.

4.2 Conjugate objects

Definition 31. Let $\mathcal{C}$ be a strict C*-tensor category and U be an object in $\mathcal{C}$. An object $\bar{U}$ is said to be a conjugate object to U if there exist morphisms $R\colon \mathbb{1} \to \bar{U} \otimes U$ and $\bar{R}\colon \mathbb{1} \to U \otimes \bar{U}$ such that

$$(\bar{R}^* \otimes \iota)(\iota \otimes R) = \iota \text{ and } (R^* \otimes \iota)(\iota \otimes \bar{R}) = \iota.$$

This identities are called the conjugate equations.

A strict C*-tensor category $\mathcal{C}$ is said to be rigid if every object in $\mathcal{C}$ has a conjugate object.

Example 32. Let H be an object in Hilb_f and $\{e_i\}_i$ be an orthonormal basis in H. Define morphisms

$$r\colon \mathbb{C} \to \bar{H} \otimes H,\ r(1) = \sum_i \bar{e}_i \otimes e_i, \tag{3}$$

$$\bar{r}\colon \mathbb{C} \to H \otimes \bar{H},\ \bar{r}(1) = \sum_i e_i \otimes \bar{e}_i. \tag{4}$$

Then the morphisms $(r, \bar{r})$ is a solution of the conjugate equations for H and $\bar{H}$. Hence $\bar{H}$ is a conjugate object to H. The morphisms $(r, \bar{r})$ do not depend on the choice of an orthonormal basis of H.

Example 33. Let U be an object in $\mathrm{Rep}\,G$. Then the conjugate representation $\bar{U} \in B(\bar{H}_U) \otimes C(G)$ is the conjugate object to U. Let r and $\bar{r}$ be the morphisms in Example 32. Then $\bar{r}$ belongs to $\mathrm{Mor}(\mathbb{1}_{\mathrm{Rep}\,G},\ U \times U^c)$. Using the operators $\rho_U \in \mathrm{Mor}(U, U^{cc})$ and $j\colon B(H) \to B(\bar{H})$, define morphism

$$R = (1 \otimes j(\rho_{\bar{U}})^{1/2})r = (1 \otimes \rho_U^{-1/2})r \in \mathrm{Mor}(\mathbb{1}, \bar{U} \times U), \tag{5}$$

$$\bar{R} = (1 \otimes j(\rho_U)^{1/2})\bar{r} = (\rho_U^{1/2} \otimes 1)\bar{r} \in \mathrm{Mor}(\mathbb{1}, U \times \bar{U}). \tag{6}$$

Then $(R, \bar{R})$ is a solution for the conjugate equations.

Theorem 34 (Frobenius reciprocity)**.** Let $\mathcal{C}$ be a C*-tensor category, $\bar{U}$ be a conjugate object to an object U in $\mathcal{C}$, and $(R, \bar{R})$ be a solution of the conjugate equations for U and $\bar{U}$. Then we have

$$\mathrm{Mor}(U \otimes V, W) \simeq \mathrm{Mor}(V, \bar{U} \otimes W),$$

and

$$\mathrm{Mor}(V \otimes U, W) \simeq \mathrm{Mor}(V, W \otimes \bar{U})$$

for any objects V, W in $\mathcal{C}$.

Proof. A map from $\mathrm{Mor}(U \otimes V, W)$ to $\mathrm{Mor}(V, \bar{U} \otimes W)$ defined by $T \mapsto (\iota_{\bar{U}} \otimes T)(R \otimes \iota_V)$ for every element T in $\mathrm{Mor}(U \otimes V, W)$ is a linear isomorphism. Its inverse map is given by $S \mapsto (\bar{R}^* \otimes \iota_W)(\iota_U \otimes S)$ for any S in $\mathrm{Mor}(V, \bar{U} \otimes W)$. Similarly, we can construct a linear isomorphism for $\mathrm{Mor}(V \otimes U, W) \simeq \mathrm{Mor}(V, W \otimes \bar{U})$. □

Corollary 35. Let $\mathcal{C}$ be a C*-tensor category, U be a simple object in $\mathcal{C}$, and $\bar{U}$ be a conjugate object to U. Then $\bar{U}$ is simple. The dimension of the space $\mathrm{Mor}(\mathbb{1}, \bar{U} \otimes U)$ and $\mathrm{Mor}(\mathbb{1}, U \otimes \bar{U})$ is equal to 1.

Proof. By the Frobenius reciprocity, one can check that the spaces

$$\mathrm{Mor}(\mathbb{1}, \bar{U} \otimes U) = \mathrm{End}(\bar{U}) = \mathrm{Mor}(\mathbb{1}, U \otimes \bar{U})$$

are isomorphic to $\mathrm{End}(U) = \mathbb{C} \cdot 1$. □

Proposition 36. Let $\mathcal{C}$ be a C*-tensor category and U be an object in $\mathcal{C}$. Assume that U has a conjugate object. Then the space $\mathrm{End}(U)$ is finite dimensional.

Proof. We give a proof by showing that there exists a positive linear functional f on End(U) such that $T \le f(U)$ for all positive element T in End(U), which is possible only for finite dimensional C*-algebra.

Define an operator $\rho_U \colon \operatorname{End}(U) \to \operatorname{End}(\mathbb{1})$ by $\rho_U(T) = R^*(\iota \otimes T)R$ for all T in End(U), where $(R, \bar{R})$ is a solution of the conjugate equations for U and $\bar{U}$. For an element $X \in \operatorname{End}(U)$ we denote Y for a morphism $(\iota \otimes X)R \in \operatorname{Mor}(\mathbb{1}, U \otimes \bar{U})$. Then there is an inequality

$$X^*X \le \|\bar{R}\|^2(\iota \otimes Y^*Y) = \|\bar{R}\|^2(\iota \otimes \rho_U(X^*X)).$$

Therefore, the functional f defined by $\|\bar{R}\|^2\rho_U(T) = f(T)1$ satisfies that $f(T)1 \le T$ for all positive T in End(T). □

Corollary 37. Every object with a conjugate decomposes into a finite direct sum of simple objects.

Definition 38. Let $\mathcal{C}$ be a strict C*-tensor category and q be a nonzero real number such that $\|q\| \le 1$. A pair (x, R) is called q-fundamental solution if x is an object in $\mathcal{C}$ and R is a morphism $\mathbb{1} \to x \otimes x$ such that

$$(R^* \otimes \iota)(\iota \otimes R) = -sgn(q)\iota_x \text{ and } R^*R = [\![2]\!]_q \iota_{\mathbb{1}},$$

where $[\![n]\!]_q$ denotes the absolute value of the q-integer

$$[n]_q = \frac{q^{-n} - q^n}{q^{-1} - q}.$$

Two q-fundamental solutions (x, R) and (y, S) in $\mathcal{C}$ are called equivalent if there exists a unitary morphism $T \in \operatorname{Mor}(x, y)$ such that $S = (T \otimes T)R$.

Definition 39. Let $\mathcal{C}$ be a C*-tensor category, U be a simple object in $\mathcal{C}$, $\bar{U}$ be a conjugate object to U, and $(R, \bar{R})$ be a solution of the conjugate equations for U and $\bar{U}$. Then the number

$$d_i(U) = \|R\| \cdot \|\bar{R}\|$$

is called the intrinsic dimension of a simple object U.

Any other solution $(R', \bar{R}')$ of the conjugate equations for U and $\bar{U}$ is of the form

$$R' = \lambda R, \ \bar{R}' = \lambda^{-1} \bar{R}$$

for some λ in $\mathbb{C}^*$. In particular, $d_i(U) = \|R\| \cdot \|\bar{R}\|$ is independent of the solution.

For general object V in $\mathcal{C}$, since it is a direct sum of simple object V_k, the intrinsic dimension of V is defined by

$$d_i(V) = \sum_k d_i(V_k).$$

For the unit object in $\mathcal{C}$, we have $d_i(\mathbb{1}) = 1$.

Example 40. Let U be an object in the category Hilb_f of finite dimensional Hilbert spaces. The intrinsic dimension of U is given by $d_i(U) = \mathrm{dim} U$, since U can be written as a direct sum of $\dim U$ objects $\mathbb{1}_{\mathrm{Hilb}_f}$.

Example 41. Let G be a compact quantum group and $\mathrm{Rep}\, G$ be a category of finite dimensional unitary representations of G. For an irreducible unitary representation U of G, by Example ? we have that the morphism $\mathrm{R} = (1 \otimes \rho_U^{-1/2}) r$ and $\bar{\mathrm{R}} = (\rho_U^{-1/2} \otimes 1) \bar{r}$ become a solution of the conjugate equations for U. Let $\{e_i\}_i$ be an orthnormal basis in the basis H_U. Then the $\| \mathrm{R} \|$ can be computed as

$$\| \mathrm{R} \| = \|(1 \otimes \rho_U^{-1/2}) r(1)\| = \| \sum_i \bar{e_i} \otimes \rho_U^{-1/2} e_i \| = \mathrm{Tr}(\rho_U^{-1})^{-1/2} = (\dim_q U)^{1/2}.$$

Similarly we can get $\|\bar{\mathrm{R}}\| = (\dim_q U)^{1/2}$. Hence the intrinsic dimension of U is $d_i(U) = \dim_q U$.

There is a multiplicativity of the intrinsic dimension on tensor product, that is,

$$d_i(U \otimes V) = d_i(U) d_i(V)$$

for objects U, V in a C*-tensor category $\mathcal{C}$.

We can give a characterization of the intrinsic dimension. For every onject U in a C*-tensor category $\mathcal{C}$. Then

$$d_i(U) = \min\{\| \mathrm{R} \| \cdot \| \bar{\mathrm{R}} \|\},$$

where the minimum is taken over all solutions of conjugate equations for U.

Next we construct a contravariant functor from the operation taking conjugates. For every object U in $\mathcal{C}$, fix a conjugate object $\bar{U}$ and a solution $(R, \bar{R})$ of the conjugate equations for $\bar{U}$. By the Frobenius reciprocity, there exists a linear isomorphism $\mathrm{Mor}(U, V) \to \mathrm{Mor}(\bar{V}, \bar{U})$, $T \mapsto T^\vee$ which is uniquely defined by

$$(\iota \otimes T)R_U = (T^\vee \otimes 1)R_V.$$

Explicitly, it is given by $T^\vee = (\iota \otimes \bar{R}_V^*)(\iota \otimes T \otimes \iota)(R_U \otimes \iota)$. This element $T^\vee$ is also defined by the identity $\bar{R}_V^*(T \otimes \iota) = \bar{R}_U^*(\iota \otimes T^\vee)$.

Proposition 42. The maps $U \mapsto \bar{U}$ and $T \mapsto T^\vee$ define a contravariant functor $\mathcal{C} \to \mathcal{C}$. When all solutions used to define $T^\vee$ are standard, then the functor is unitary and its square is naturally unitarily isomorphic to the identity functor.

Proof. Since $(ST)^\vee = T^\vee S^\vee$, this functor is a contravariant functor. When all solutions to define $T^\vee$ are standard, then in order to show that $T^{\vee *} = T^{*\vee}$ for any $T \in \mathrm{Mor}(U, V)$, it is enough to proov that $\mathrm{Tr}_{\bar{U}}(ST^{*\vee}) = \mathrm{Tr}_{\bar{U}}(ST^{\vee *})$ for every $S \in \mathrm{Mor}(\bar{V}, \bar{U})$. We compute that

$$\begin{aligned}
\mathrm{Tr}_{\bar{U}}(ST^{*\vee}) &= R_U^*(ST^{*\vee} \otimes \iota)R_U = R_U^*(S \otimes T^*)R_V \\
&= ((\iota \otimes T)R_V)^*(S \otimes \iota)R_V \\
&= R_V^*(T^{\vee *}S \otimes \iota)R_V \\
&= \mathrm{Tr}_{\bar{U}}(T^{\vee *}S) = \mathrm{Tr}\,\bar{U}(ST^{\vee *}).
\end{aligned}$$

Therefore, we obtain $T^{\vee *} = T^{*\vee}$.

Since $(R_U, \bar{R}_U)$ and $(\bar{R}_U, R_U)$ are both standard solutions for $\bar{U}$, there exists a unitary morphism $\eta_U \in \mathrm{Mor}(U, \bar{\bar{U}})$ such that

$$(\iota \otimes \eta_U)R_U = \bar{R}_{\bar{U}}, \quad (\eta_V \otimes \iota)\bar{R}_U = R_{\bar{U}}.$$

For a morphism $T \in \mathrm{Mor}(U, V)$, we have

$$\begin{aligned}(\iota \otimes \eta_V T)R_U &= (T^\vee \otimes \eta_V)R_V = (T^\vee \otimes \iota)\bar{R}_{\bar{V}} \\ &= (\bar{R}_V^*(T^{\vee *} \otimes \iota))^* \; = (\bar{R}_{\bar{V}}^*(\iota \otimes T^{\vee * \vee}))^* \\ &= (\iota \otimes T^{\vee * \vee *})\bar{R}_{\bar{U}} = (\iota \otimes T^{\vee * \vee *}\eta_U)R_U.\end{aligned}$$

It shows that $\eta_V T = T^{\vee * \vee *}\eta_U$. Using the identity $T^{\vee *} = T^{*\vee}$, we get $\eta_V T = T^{\vee\vee}\eta_U$. Hence the unitaries η_U defines a natural isomorphism between the identity functor and the functor $U \mapsto \bar{\bar{U}}$. $\square$

We remark that for any solutions $(R_U, \bar{R}_U)$ for U and $(R_V, \bar{R}_V)$ for V, if we put $\bar{R}_{\bar{U}} = R_U$, $\bar{R}_{\bar{V}} = R_V$ then we get $T^{\vee * \vee *} = T$ for every element T in $\mathrm{Mor}(U, V)$.

A contravariant functor $F \colon \mathcal{C} \to \mathcal{C}$ given by $F(U) = \bar{U}$, $F(T) = T^\vee$ can be a tensor functor by giving $F_2(U, V) \in \mathrm{Mor}(\bar{V} \otimes \bar{U}, \overline{U \otimes V})$ as

$$(F_2(U, V) \otimes \iota \otimes \iota)(\iota \otimes R_U \otimes \iota)R_V = R_{U \otimes V}.$$

4.3 Fiber functors

Definition 43. Let $\mathcal{C}$ be a C*-tensor category, F be a tensor functor from $\mathcal{C}$ to the category $Hilb_f$ of finite dimensional Hilbert spaces. The functor F is said to be a fiber functor if it is faithful (that is, injective on morphism) and exact.

When $\mathcal{C}$ is a C*-tensor category with conjugates, then any object U in $\mathcal{C}$ is a direct sum of simple objects U_k. In this case, any linear functor

$\mathcal{C} \to \mathrm{Hilb}_f$ is exact. Furthermore, if U is nonzero, then the unit object $\mathbb{1}_{\mathcal{C}}$ is a subobject of $\bar{U} \otimes U$. Thus for a tensor functor $F \colon \mathcal{C} \to \mathrm{Hilb}_f$ we have $\mathbb{C} \subseteq F(\bar{U} \otimes U) \simeq F(\bar{U}) \otimes F(U)$. It leads that $F(U)$ is also nonzero. By the fact that a linear functor is faithful if and only if the image of every simple object is nonzero, the functor F is fiber functor. In short, for C*-tensor categories with conjugates a fiber functor is just a tensor functor $\mathcal{C} \to \mathrm{Hilb}_f$.

Example 44. Let G be compact quantum group. Define a tensor functor $F \colon \mathrm{Rep}\, G \to \mathrm{Hilb}_f$ by $F(U) = H_U$ for every finite dimensional unitary representation U of G, and the action of F on morphisms and F_2 to be dentity maps. Then F is a unitary fiber functor and it is called a canonical fiber functor on $\mathrm{Rep}\, G$.

Theorem 45 (Woronowicz's Tannaka–Krein duality)**.** Let $\mathcal{C}$ be a C*-tensor category with conjugates, $F \colon \mathcal{C} \to \mathrm{Hilb}_f$ be a unitary fiber functor. Then there exists a compact quantum group G and a unitary monoidal equivalence $E \colon \mathcal{C} \to \mathrm{Rep}\, G$ such that F is naturally unitarily monoidally isomorphic on $\mathrm{Rep}\, G$. Furthermore, the Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$ for such G is uniquely determined up to isomorphisms.

For the proof of the above theorem, we may assume that $\mathcal{C}$ is strict, $F(\mathbb{1}_{\mathcal{C}}) = \mathbb{C}$, and $F_0 = \mathrm{id}$. Consider a $*$-algebra $\mathrm{End}(F) = \mathrm{Nat}(F, F)$ of natural transformations from F to F. Let U be the representatives of isomorphic classes of simple objects in $\mathcal{C}$. Then, since every object U in $\mathcal{C}$ can be expressed as a direct sum of U_α, a natural transformation $\eta \colon F \to F$ is completely determined by morphisms $\eta_{U_\alpha} \colon F(U_\alpha) \to F(U_\alpha)$ in Hilb_f. So the $*$-algebra $\mathrm{End}(F)$ can be identified with $\prod_\alpha B(F(U_\alpha))$.

Next, using the tensor functor structure on F, we define a $*$-homomorphism δ from $\mathrm{End}(F)$ to $\mathrm{End}(F^{\otimes 2}) \simeq \prod_{\alpha,\beta} B(F(U_\alpha) \otimes F(U_\beta))$ such that $\delta(\eta)$ is determined by the commutative diagram that means $\delta(\eta)_{U,V} = F_2^* \eta_{U \otimes V} F_2$.

Now we extend the $*$-homomorphisms $(\iota \otimes \delta)$, $(\delta \otimes \iota)$ to $\mathrm{End}(F^{\otimes 2}) \to$

$$\begin{array}{ccc} F(U)\otimes F(V) & \xrightarrow{\delta(\eta)_{U,V}} & F(U\otimes V) \\ \downarrow F_2 & & \downarrow F_2 \\ F(U)\otimes F(V) & \xrightarrow{\eta_{U\otimes V}} & F(U\otimes V) \end{array}$$

$\mathrm{End}(F^{\otimes 3})$ by the definition of a tensor functor. Then, the identity $(\iota\otimes\delta)\delta = (\delta,\otimes\iota)\delta$ holds.

In the following, we would like to show that $(\mathrm{End}(F),\Delta)\simeq(\mathcal{U}(G),\hat{\Delta})$ for some compact quantum group G. For a morphism T from an object U to an object $V\otimes W$ in a C*-tensor category $\mathcal{C}$, denote by $\Theta(T)$ for the morphism determined by

$$\Theta(T) = F_2^* F(T) \colon F(U) \to F(V)\otimes F(W)$$

in Hilb_f. If $(R,\bar{R})$ is a solution of the conjugate equations for U and a cojugate object $\bar{U}$ to U, then $(\Theta(R),\Theta(\bar{R}))$ is a solution of the conjugate equations for $F(U)$ and $F(\bar{U})$ in Hilb_f.

Lemma 46. Let $\mathcal{C}$ be a C*-tensor category with conjugates, $F\colon\mathcal{C}\to\mathrm{Hilb}_f$ be a unitary fiber functor, and $\mathrm{End}(F)$ be a $*$-algebra of natural transformations from F to itself. For every element η in $\mathrm{End}(F)$, there exists a unique element $\eta^\vee$ in $\mathrm{End}(F)$ such that if $(R,\bar{R})$ is a solution of the conjugate equations for U and $\bar{U}$ in $\mathcal{C}$, then we have

$$(\eta^\vee)_{\bar{U}} = (\eta_U)^\vee,$$

where $(\eta_U)^\vee$ is defined using the solution $(\Theta(R),\Theta(\bar{R}))$ of the conjugate equations for $F(U)$ and $F(\bar{U})$.

Proof. For an element $\eta_U\in\mathrm{End}(F)$, the map $\eta_U\mapsto(\eta_U)^\vee$ is the linear isomorphism between $\mathrm{End}(U)$ and $\mathrm{End}(\bar{U})$ defined by

$$(\iota\otimes\eta_U)R = ((\eta_U)^\vee\otimes\iota)R.$$

For a fixed object $\bar{U}$, the morphism $(\eta_U)^\vee$ does not depend on $(R, \bar{R})$. Indeed, when $(R', \bar{R}')$ is any other solution of the conjugate equations for U and $\bar{U}$, then using a morphism T from U to $\bar{U}$ we have $R' = (\iota \otimes T^*)R$ and $\bar{R}' = (T^{-1} \otimes \iota)\bar{R}$. Thus the equation

$$\Theta(R') = (1 \otimes F(T^*))\Theta(R)$$

holds. By the naturality of η, we have that $\eta_U F(T^*) = F(T^*)\eta_U$. It leads that $(\eta_U)^\vee$ is independent whether we use $\Theta(R)$ or $\Theta(R')$ to define it.

By the naturality of η, we can check that for two isomorphic objects U and U' such that their conjugate objects are same, then $(\eta_U)^\vee = (\eta_{U'})^\vee$. Therefore, there is an well-defined collection of maps $(\eta^\vee)_V \colon F(U) \to F(V)$ such that if V is a conjugate object to U, then $(\eta^\vee)_V = (\eta_U)^\vee$. In the following we check that the maps $(\eta^\vee)_V$ are natural. Consider an operator $S \colon V_1 \to V_2$ such that $S = T^\vee$ for some $T \colon U_2 \to U_1$, where we use fixed solutions $(R_1, \bar{R}_1)$, $(R_2, \bar{R}_2)$ of the conjugate equations for (U_1, V_1) and (U_2, V_2). By the identity $\eta_{U_1} F(T) = F(T)\eta_{U_2}$, we get

$$F(T)^\vee(\eta_{U_1})^\vee = (\eta_{U_2})^\vee F(T)^\vee,$$

where we use $\Theta(R_1), \Theta(R_2)$ to define $F(T)^\vee$. Using the identity $F(T)^\vee = F(T^\vee) = F(S)$, we obtain $F(S)(\eta^\vee)_{V_1} = (\eta^\vee)_{V_2} F(S)$. $\square$

Now we are ready to define (A, Δ) for $(\mathbb{C}[G], \Delta)$. From the identification $\mathrm{End}(F) \simeq \prod_\alpha B(F(U_\alpha))$, we define $A = \bigoplus_\alpha B(F(U_\alpha))^* \in \mathrm{End}(F)^*$. For two elements a, b in A, their tensor product $a \otimes b \in \mathrm{End}(F^{\otimes 2})^*$ is well-defined. Thus we define a product on A by

$$ab = (a \otimes b)\delta.$$

If a is in $B(F(U_\alpha))^*$ and b is in $B(F(U_\beta))^*$, then ab is an element of $\bigoplus_\gamma B(F(U_\gamma))^*$, with finitely many γ such that $\mathrm{Mor}(U_\gamma, U_\alpha \otimes U_\beta) \neq 0$. Since δ is coassociative, the operation $a, b \mapsto a \otimes b$ is associative. The algebra A is unital and its unit is given by $1_A(\eta) = \eta_{\mathbb{1}} \in \mathrm{End}(\mathbb{C}) = \mathbb{C}$. Define a coproduct $\Delta\colon A \to A \otimes A$ by

$$\Delta(a)(\omega \otimes \eta) = a(\omega\eta)$$

for ω, η in $\mathrm{End}(F)$ and $a \in A$. This is a unital coassociative homomorphism. Define a character $\epsilon\colon A \to \mathbb{C}$ by

$$\epsilon(a) = a(1_{\mathrm{End}(F)}) \tag{7}$$

and a linear map $S\colon A \to A$ by

$$S(a)(\eta) = a(\eta^\vee). \tag{8}$$

Lemma 47. A pair (A, Δ) of the algebra A and the coproduct Δ defined as above is a Hopf $*$-algebra with the counit ϵ and the antipode S defined in (7) and (8).

For the proof of the lemma, we use the following convention. We regard $\delta(\eta)$ as a finite sum of elementary tensors so that we omit the sum symbol and write simply as $\delta(\eta) = \eta_{(1)} \otimes \eta_{(2)}$. This is called the Sweedler's sumless notation.

Proof. The identities $(\iota \otimes \epsilon)\Delta(a) = a$ and $(\epsilon \otimes \iota)\Delta(a) = a$ can be checked by applying $\eta \in \mathrm{End}(F)$ to the both sides. It is known that the identity $m(\iota \otimes S)\Delta(a) = \epsilon(a)1$ for all a in A holds if and only if

$$\eta_{(1)}\eta_{(2)}^\vee = \eta_{\mathbb{1}_\mathcal{C}} 1_{\mathrm{End}(F)} \tag{9}$$

for every η in $\mathrm{End}(F)$.

Fix an object U in $\mathcal{C}$ and a solution $(R, \bar{R})$ of the conjugate equation for U and $\bar{U}$. By the definition $(1 \otimes T)\Theta(R) = (T^\vee \otimes 1)\Theta(R)$ of $\vee$, we get

$$\begin{aligned}((\eta_{(1)})_{\bar{U}}(\eta_{(2)}^\vee)_{\bar{U}} \otimes \iota)\Theta(R) &= \delta(\eta)_{\bar{U},U}\Theta(R)\\ &= F_2^*\eta_{\bar{U}\otimes U}F(R)\\ &= \Theta(R)\eta_{\mathbb{1}}.\end{aligned}$$

Therefore, the identity $(\eta_{(1)}\eta^{\vee}_{(2)})_{\bar{U}} = \eta_{\mathbb{1}} 1_{\mathrm{End}(F)}$ holds in $B(F(\bar{U}))$. This is true for all U in $\mathcal{C}$, so the equation (9) holds. Similarly, by $\Theta(R)^*(T \otimes 1) = \Theta(\bar{R})^*(1 \otimes T^{\vee})$ for $\vee$, we can get $m(S \otimes \iota)\Delta(a) = \epsilon(a)1$.

Define an antilinear map $a \mapsto a^*$ by $a^*(\eta) = \bar{a}(\eta^{\vee}) = \overline{a(\eta^*)}$. Then, using the property $\eta^{\vee * \vee *} = \eta$ we have $a^{**} = a$. The coproduct Δ is $*$-preserving. The map $*$ is anti-multiplicative since it is equivalent to anti-multiplicativity of the antipode S on A and this is true for any Hopf algebra. Therefore, (A, Δ) is a Hopf $*$-algebra. □

For any object U in $\mathcal{C}$ define $X^U \in B(F(U)) \otimes \mathrm{End}(F)^*$ by

$$(\iota \otimes \eta)(X^U) = \eta_U$$

for every $\eta \in \mathrm{End}(F)$. Clearly X^U is in $B(F(U)) \otimes A$.

Lemma 48. Let (A, Δ) be the Hopf $*$-algebra defined as above. Then we have:

1. The element $X^U \in B(F(U)) \otimes A$ is a unitary corepresentation of (A, Δ).
2. When T is an element in $Mor(U, V)$, then the identity $(F(T) \otimes 1)X^U = X^V(F(T) \otimes 1)$ holds.
3. $(F_2 \otimes 1)X^U_{13}X^U_{23} = X^{U \otimes V}(F_2 \otimes 1)$.

Proof. (i) The identity $(\iota \otimes \epsilon)(X^U) = 1$ follows from the definition of X^U since $\epsilon = 1 \in \mathrm{End}(F)$. The identity $(\iota \otimes \Delta)(X^U) = X^U_{12}X^U_{13}$ is checked by applying $(\iota \otimes \omega \otimes \eta)$ to both sides. Thus, X^U is a coprepresentation of (A, Δ). It leads that X^U is invertible and $(\iota \otimes S)(X^U) = (X^U)^{-1}$. Hence, for every η in $\mathrm{End}(F)$,

$$\begin{aligned}(\iota \otimes \eta)((X^U)^{-1}) &= (\iota \otimes \eta^{\vee})(X^U) = (\eta^{\vee})_U = (\eta^{\vee *})^*_U \\ &= (\iota \otimes \eta^{\vee *})(X^U) = (\iota \otimes \eta)((X^U)^*).\end{aligned}$$

Therefore, we get $(X^U)^{-1} = (X^U)^*$.

(ii) By applying $(\iota \otimes \eta)$ to both sides, the equation can be checked using the naturality of η.

(iii) Applying $(\iota \otimes \iota \otimes \eta)$ to the left hand side we get

$$\begin{aligned} F_2(\iota \otimes \iota \otimes \eta)(X_{13}^U X_{23}^V) &= F_2(\iota \otimes \iota \otimes \delta(\eta))(X_{13}^U X_{24}^V) \\ &= F_2\delta(\eta)_{U,V} = \eta_{U\otimes V} F_2. \end{aligned}$$

This is equal to $(\iota \otimes \iota \otimes \eta)X^{U\otimes V}(F_2 \otimes 1)$. □

Now we are ready to give a proof of Woronowicz's theorem.

Proof of Woronowicz's theorem. By Theorem, we have $(A, \Delta) = (\mathbb{C}[G], \Delta)$ for a compact quantum group G. By Lemma 48 (ii) and (iii), we can define a unitary tensor functor $E\colon \mathcal{C} \to \operatorname{Rep} G$ by $E(U) = X^U$ for every object U in $\mathcal{C}$, $E(T) = F(T)$ for morphism T, and $E_2 = F_2$. Then, the functor $F\colon \mathcal{C} \to \mathrm{Hilb}_f$ is the composition of E with the canonical fiber functor on $\operatorname{Rep} G$. The representation X^{U_α} of G are irreducible, pairwise nonequivalent, and form equivalence classes of irreducible representation of G, since the matrix coeeficients of X^{U_α} form a basis in $\bigoplus B(F(U_\alpha))^* = A$. Thus, the functor E is a unitary monoidal equivalence.

To show the uniqueness, let G' be a compact quantum group and F' be the canonical fiber functor $\operatorname{Rep} G' \to \mathrm{Hilb}_f$ on $\operatorname{Rep} G'$. Then $(\mathcal{U}(G'), \hat{\Delta}) \simeq (\operatorname{End}(F'), \delta)$. By seeing $\omega \in \mathcal{U}(G')$ as an element $\operatorname{End}(F')$, it acts on $F'(U) = H_U$ by $\pi_U(\omega)$. Let $E'\colon \mathcal{C} \to \operatorname{Rep} G'$ be a unitary monoidal equivalence such that $F'E'$ is naturally unitarily monoidally isomorphic to F. Then we have an isomorphism

$$(\mathcal{U}(G), \hat{\Delta}) = (\operatorname{End}(F), \delta) \simeq (\operatorname{End}(F'), \delta) = (\mathcal{U}(G'), \hat{\Delta}).$$

By duality we obtain $(\mathbb{C}[G], \Delta) = (\mathbb{C}[G'], \Delta)$. □

Definition 49. Let G_1, G_2 be compact quantum groups. Then G_1 are G_2 are said to be monoidally equivalent if $\operatorname{Rep} G_1$ and $\operatorname{Rep} G_2$ are unitarily monoidally equivalent.

Example 50. The two free orthogonal compact quantum groups O_F^+ and $O_{F'}^+$ are monoidally equivalent if and only if the identities $\operatorname{sgn}(F\bar{F}) = \operatorname{sgn}(F'\bar{F}')$ and $\operatorname{Tr}(F^*F) = \operatorname{Tr}(F'^*F')$ hold.

The two free orthogonal compact quantum groups O_F^+ and $O_{F'}^+$ are isomorphic if and only if the matrices F and F' are same size and $F' = vFv^t$ for some unitary matrix v.

4.4 Module C*-categories

Definition 51 (e.g., [3])**.** Let $\mathcal{D}$ be a C*-category. and $\mathcal{C}$ be a C*-tensor category. Then $(\mathcal{D}, M, \phi, e)$ is a left $\mathcal{C}$-module C*-category if M is a bilinear $*$-functor $\mathcal{C}\times\mathcal{D} \to \mathcal{D}$ with natural unitary transformations $\phi\colon M((-\otimes-),-) \to M(-, M(-,-))$ and $e\colon M(\mathbb{1},-) \to \mathrm{id}$ satisfying certain coherence condition. We often abbreviate this left $\mathcal{C}$-module C*-category as $\mathcal{D}$.

When we write $U \otimes X$ for $M(U,X)$, then the condition is described as the commutative diagrams below.

$$\begin{array}{ccc}
(U \otimes V \otimes W) \otimes X & \xrightarrow{\phi_{U,V\otimes W,X}} & U \otimes ((V \otimes W) \otimes X) \\
\Big\downarrow {\scriptstyle \phi_{U\otimes V,W,X}} & & \Big\downarrow {\scriptstyle \mathrm{id}_U \otimes \phi_{V,W,X}} \\
(U \otimes V) \otimes (W \otimes X) & \xrightarrow{\phi_{U\otimes V,W,X}} & U \otimes (V \otimes (W \otimes X))
\end{array}$$

$$\begin{array}{ccccc}
 & \overset{\phi_{U,\mathbb{1},X}}{\nearrow} & U \otimes (\mathbb{1} \otimes X) & \overset{\mathrm{id}_U \otimes e_X}{\searrow} & \\
U \otimes X & & \xrightarrow{\mathrm{id}_{U\otimes X}} & & U \otimes X \\
 & \underset{\phi_{\mathbb{1},U,X}}{\searrow} & \mathbb{1} \otimes (U \otimes X) & \underset{e_{U\otimes X}}{\nearrow} &
\end{array}$$

Example 52. Let G be a compact quantum group,and H be a closed quantum subgroup of G. Then the category $\operatorname{Rep} H$ of finite dimensional unitary representation of H is a $\operatorname{Rep} G$-module C*-category. For $\pi \in \operatorname{Rep} G$ and $\theta \in H$, $\pi \otimes \theta$ is defined by $\pi|_H \otimes \theta$. The restriction functor $\operatorname{Rep} G \to \operatorname{Rep} H$ induces this module category. We are particularly interested in the case of $G = SU_{-1}(2)$ and $H = G_{\mathrm{KP}}$.

Definition 53. Let $\mathcal{D}$ and $\mathcal{D}'$ be module C*-categories over a C*-tensor category $\mathcal{C}$. Then a $\mathcal{C}$-module homomorphism (G, ψ) is from $\mathcal{D} \to \mathcal{D}'$ is a C*-tensor functor $G\colon \mathcal{D} \to \mathcal{D}'$ and a natural unitary equivalence $\psi\colon G(-\otimes-) \to -\otimes G-$ satisfying the commutative diagrams below.

$$
\begin{array}{ccc}
G(\mathbb{1} \otimes X) & \xrightarrow{\psi_{\mathbb{1},X}} & \mathbb{1} \otimes GX \\
{\scriptstyle G(e)}\downarrow & \swarrow{\scriptstyle e} & \\
GX & &
\end{array}
\qquad
\begin{array}{ccccc}
 & & U \otimes G(V \otimes X) & & \\
 & \nearrow{\scriptstyle \psi_{U,V\otimes X}} & & \searrow{\scriptstyle \mathrm{id}_U \otimes \psi_{V,X}} & \\
G(U \otimes (V \otimes X)) & & & & U \otimes (V \otimes GX) \\
{\scriptstyle G(\phi_{U,V,X})}\downarrow & & & & \downarrow{\scriptstyle \phi_{U,V,GX}} \\
G((U \otimes V) \otimes X) & & \xrightarrow[\psi_{U\otimes V,X}]{} & & (U \otimes V) \otimes GX
\end{array}
$$

A $\operatorname{Rep} G$-module homomorphism for a compact quantum group G corresponds to the Hopf homomorphism which define the action of G.

Definition 54. Let X be a quantum homogeneous space for a compact quantum group G. An equivariant Hilbert C*-module $\mathcal{E}$ over X is a right Hilbert $C(X)$-module $\mathcal{E}$, carrying a coaction $\alpha_{\mathcal{E}}\colon \mathcal{E} \to \mathcal{E} \otimes C(G)$, where the right hand side is the exterior product of $\mathcal{E}$ with the standard right Hilbert $C(G)$-module $C(G)$, satisfying the density condition $[(1 \otimes C(G))\alpha_{\mathcal{E}}(\mathcal{E})]^{\text{n-cl}} = \mathcal{E} \otimes C(G) = [\alpha_{\mathcal{E}}(\mathcal{E})(1 \otimes C(G))]^{\text{n-cl}}$ and the compatibility conditions

1. $\alpha_{\mathcal{E}}(\xi \cdot x) = \alpha_{\mathcal{E}}(\xi)\alpha_X(x)$ for all $x \in C(X)$ and $\xi \in \mathcal{E}$;

2. $\langle \alpha_{\mathcal{E}}(\xi), \alpha_{\mathcal{E}}(\eta)\rangle_{C(X)\otimes C(G)} = \alpha_X(\langle \xi, \eta\rangle_{C(X)})$ for all $\xi, \eta \in \mathcal{E}$.

An equivariant Hilbert C*-module $\mathcal{E}$ is called finite if $\mathcal{E}$ is finitely generated projective as a right C*-module, and irreducible if $\mathcal{L}_G(\mathcal{E}) = \{T \in \mathcal{L}(\mathcal{E}) | \alpha_{\mathcal{E}}(T\xi) = (T \otimes 1)\alpha_\xi$ for any $T \in \mathcal{E}\}$ is one-dimensional.

A category $\mathcal{D}_X$ of finite equivariant Hilbert C*-modules over X with the equivariant adjointable maps between Hilbert C*-modules is a semi-simple C*-category. Then by the operation $\mathrm{Rep}(G) \times \mathcal{D}_x \to \mathcal{D}, (u, \mathcal{E}) \mapsto u \otimes \mathcal{E}$, it become a connected Rep(G)-module C*-category.

5 Tambara–Yamagami tensor category and Kac–Paljutkin Hopf algebra

5.1 Tambara–Yamagami tensor category

One of the Tambara–Yamagami tensor categories [15] arising from the Klein 4-group $K_4 = \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is realised as the category of representations of the Kac–Paljutkin Hopf algebra [15]. Let us recall that the elements in the notation in [15] of $K_4 = \{e, a, b, c\}$ satisfies the relations $a^2 = b^2 = c^2 = e,\ ab = c,\ bc = a, ca = b$. What we focus on is the tensor category $\mathcal{C}(\chi, \tau)$ corresponding to the nondegenerate symmetric bicharacter $\chi = \chi_c$ of K_4 which is defined by

$$\chi_c(a, a) = \chi_c(b, b) = -1,\ \chi_c(a, b) = 1,$$

and the parameter $\tau = \frac{1}{2}$ satisfying $\tau^2 = \frac{1}{|K_4|}$. Its objects are finite direct sums of elements in $S = K_4 \cup \{\rho\}$. Sets of morphisms between elements in S are given by

$$\mathrm{Mor}(s, s') = \begin{cases} \mathbb{C} & s = s', \\ 0 & s \neq s', \end{cases}$$

so S is the set of irreducible classes of $\mathcal{C}(\chi, \tau)$. Tensor products of elements in S are given by

$$s \otimes t = st,\ s \otimes \rho = \rho = \rho \otimes s,\ \rho \otimes \rho = \bigoplus_{s \in K_4} s,\ (s, t \in K_4)$$

and the unit object is e. Associativity morphisms $\varphi_{U,V,W} \colon (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ are given by

$$\varphi_{s,t,u} = \mathrm{id}_{stu}, \qquad \varphi_{s,t,\rho} = \varphi_{\rho,s,t} = \mathrm{id}_\rho,$$

$$\varphi_{s,\rho,t} = \chi_c(s, t)\mathrm{id}_t, \qquad \varphi_{s,\rho,\rho} = \varphi_{\rho,\rho,s} = \bigoplus_{k \in K_4} \mathrm{id}_k,$$

$$\varphi_{\rho,s,\rho} = \bigoplus_{k \in K_4} \chi_c(s, t)\mathrm{id}_k, \quad \varphi_{\rho,\rho,\rho} = \left(\frac{1}{2}\chi_c(k, l)^{-1}\mathrm{id}_\rho\right)_{k,l} : \bigoplus_{k \in K_4} \rho \to \bigoplus_{l \in K_4} \rho,$$

for $s, t, u \in K_4$. This category is identified with the representation category of Kac–Paljutkin quantum group G_{KP}, that is

$$\mathcal{C}\left(\chi_c, \frac{1}{2}\right) \simeq \mathrm{Rep}(G_{\mathrm{KP}})$$

as tensor categories. Here $(C(G_{\mathrm{KP}}), \Delta)$ is the Kac–Paljutkin algebra, that is, the eight dimensional Hopf algebra which is the noncommutative and noncocommutative algebra. It is given by

$$C(G_{\mathrm{KP}}) = \mathbb{C} \cdot \epsilon \oplus \mathbb{C} \cdot \alpha \oplus \mathbb{C} \cdot \beta \oplus \mathbb{C} \cdot \gamma \oplus M_2(\mathbb{C}),$$

as an ($*$-)algebra. The comultiplication $\Delta\colon C(G_{\mathrm{KP}}) \to C(G_{\mathrm{KP}}) \otimes C(G_{\mathrm{KP}})$ is defined by

$$\begin{aligned}
\Delta(\epsilon) &= \epsilon \otimes \epsilon + \alpha \otimes \alpha + \beta \otimes \beta + \gamma \otimes \gamma + \frac{1}{2} \sum_{1 \le i,j \le 2} \epsilon_{ij} \otimes \epsilon_{ij}, \\
\Delta(\alpha) &= \epsilon \otimes \alpha + \alpha \otimes \epsilon + \beta \otimes \gamma + \gamma \otimes \beta \\
&\quad + \frac{1}{2}(\epsilon_{11} \otimes \epsilon_{22} + i\epsilon_{12} \otimes \epsilon_{21} - i\epsilon_{21} \otimes \epsilon_{12} + \epsilon_{22} \otimes \epsilon_{11}), \\
\Delta(\beta) &= \epsilon \otimes \beta + \beta \otimes \epsilon + \alpha \otimes \gamma + \gamma \otimes \alpha \\
&\quad + \frac{1}{2}(\epsilon_{11} \otimes \epsilon_{22} - i\epsilon_{12} \otimes \epsilon_{21} + i\epsilon_{21} \otimes \epsilon_{12} + \epsilon_{22} \otimes \epsilon_{11}), \\
\Delta(\gamma) &= \epsilon \otimes \gamma + \gamma \otimes \epsilon + \alpha \otimes \beta + \beta \otimes \alpha \\
&\quad + \frac{1}{2}(\epsilon_{11} \otimes \epsilon_{11} - \epsilon_{12} \otimes \epsilon_{12} - \epsilon_{21} \otimes \epsilon_{21} + \epsilon_{22} \otimes \epsilon_{22}), \\
\Delta(x) &= \epsilon \otimes x + \alpha \otimes u_\alpha x u_\alpha^* + \beta \otimes u_\beta x u_\beta^* + \gamma \otimes u_\gamma x u_\gamma^* \\
&\quad + x \otimes \epsilon + \overline{u}_\alpha x \overline{u}_\alpha^* \otimes \alpha + \overline{u}_\beta x \overline{u}_\beta^* \otimes \beta + \overline{u}_\gamma x \overline{u}_\gamma^* \otimes \gamma,
\end{aligned} \tag{10}$$

for projections $\epsilon, \alpha, \beta, \gamma$ and $x \in M_2(\mathbb{C})$, where ϵ_{ij} are the matrix units in $M_2(\mathbb{C})$ and

$$u_\alpha = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix}, \; u_\beta = \begin{pmatrix} 0 & 1 \\ i & 0 \end{pmatrix}, \; u_\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

5.2 Graded twisting of Hopf algebras

Let us describe the graded twist construction [2]. The algebra $C(SU(2))$ of continuous functions on the compact group $SU(2)$ can be identified with the space $\bigoplus_{n\in\mathbb{N}} \overline{H}_{n/2} \otimes H_{n/2}$, where $\overline{H}_{n/2} \otimes H_{n/2}$ denotes matrix coefficients of the irreducible representation of $SU(2)$ of dimension $n+1$. Half integers $\{n/2\}_{n\in\mathbb{N}}$ can be divided into integers $\{0,1,2,3,\ldots\}$ for n even and others $\{1/2, 3/2, 5/2, \ldots\}$ for n odd. Thus the space above can be decomposed as

$$(\bigoplus_{n:\text{ even}} \overline{H}_{n/2} \otimes H_{n/2}) \oplus (\bigoplus_{n:\text{ odd}} \overline{H}_{n/2} \otimes H_{n/2}).$$

The component with even n forms the algebra of continuous functions on $SO(3)$, so we denote the whole space by $C(SO(3)) \oplus C(SU(2))_{\text{odd}}$. Let $\{e_1, e_2\}$ be an orthonormal basis of $H_{1/2}$. Unit vectors $\overline{e}_i \otimes e_j$ in $\overline{H}_{1/2} \otimes H_{1/2} \subset C(SU(2))_{\text{odd}}$ are denoted by u_{ij}.

Consider an action α of the group $\mathbb{Z}/2\mathbb{Z}$ on the Hopf algebra $C(SU(2))$ defined by

$$\begin{aligned} \alpha_g \left[\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \right] &= \begin{pmatrix} u_{11} & -u_{12} \\ -u_{21} & u_{22} \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \end{aligned} \tag{11}$$

for the generator g of $\mathbb{Z}/2\mathbb{Z}$.

Next take the crossed product $C(SU(2)) \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}$ of the Hopf algebra. Define the graded twisting of $C(SU(2))$ by α as the subalgebra of crossed product

$$C(SU(2))^{t,\alpha} = C(SO(3)) \oplus (C(SU(2))_{\text{odd}} \cdot \lambda_g) \subset C(SU(2)) \rtimes_\alpha \mathbb{Z}/2\mathbb{Z}.$$

Generators $u_{ij}\lambda_g$ in $C(SU(2))^{t,\alpha}$ are denoted by u'_{ij}. It follows that the matrix $(u'_{ij})_{i,j=1}^2$ becomes unitary because $(u_{ij})_{i,j=1}^2$ is an unitary matrix. They satisfy the same relations as the generators of $C(SU_{-1}(2))$. Indeed,

$$(u'_{11})^* = (u_{11}\lambda_g)^* = \lambda_{g^{-1}}u^*_{11} = \lambda_g u_{22} = u_{22}\lambda_g = u'_{22},$$
$$(u'_{21})^* = (u_{21}\lambda_g)^* = \lambda_{g^{-1}}u^*_{21} = \lambda_g(-u_{21}) = u_{12}\lambda_g = u'_{12},$$

so that the matrix $(u'_{ij})^2_{i,j=1}$ can be described as

$$\begin{pmatrix} u'_{11} & u'_{12} \\ u'_{21} & u'_{22} \end{pmatrix} = \begin{pmatrix} u'_{11} & (u'_{21})^* \\ u'_{21} & (u'_{11})^* \end{pmatrix}.$$

Moreover, images of u'_{ij} by the comultiplication Δ_{gr} on $C(SU(2))^{t,\alpha}$ are given by

$$\Delta_{gr}(u'_{ij}) = (\sum_k u_{ik} \otimes u_{kj})\lambda_g \otimes \lambda_g = \sum_k u'_{ik} \otimes u'_{kj} \tag{12}$$

Thus we obtain an isomorphism of Hopf algebras

$$C(SU(2))^{t,\alpha} \simeq C(SU_{-1}(2))$$

by mapping u'_{ij} in $C(SU(2))^{t,\alpha}$ to $u^{(-1)}_{ij}$ in $C(SU_{-1}(2))$.

A similar construction works for any $\mathbb{Z}/2\mathbb{Z}$-graded Hopf algebra with grade-preserving action of $\mathbb{Z}/2\mathbb{Z}$.

6 Realization of Kac–Paljutkin Hopf algebra as a quotient

Besides the formulation of graded twist, [2] provided a method for describing quantum subgroups of a compact quantum group obtained as the graded twisting of a genuine compact group. Let us apply their method to our compact quantum group $SU_{-1}(2)$.

Proposition 55 ([2, Example 4.11])**.** Any quantum subgroup of $SU_{-1}(2)$ with noncommutative function algebra corresponds to a closed subgroup of $SU(2)$ containing $\{\pm I_2\}$, being stable under the $\mathbb{Z}/2\mathbb{Z}$-action defined in (11) and containing an element

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ with } abcd \neq 0. \tag{13}$$

Take the subgroup $\tilde{V}$ of $SU(2)$ generated by

$$s_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} i & i \\ i & -i \end{pmatrix}, \; s_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ i & i \end{pmatrix}, \; s_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

They satisfy relations $s_i^2 = -I_2$ for $i = 1, 2, 3$ and $s_1 s_2 = s_3 = -s_2 s_1$. It is the group of order eight with elements $\pm s_i$ for $i = 1, 2, 3$ and $\pm I_2$. This subgroup $\tilde{V}$ is related to the Klein 4-group K_4 via an isomorphism $\tilde{V}/\{\pm I_2\} \simeq K_4$. This subgroup satisfies the conditions mentioned in the above proposition. Namely, $s_i^2 = -I_2$ for $i = 1, 2, 3$ so $\tilde{V}$ has the elements $\pm I_2$. Since α_g transforms the elements

$$s_1 \mapsto -s_2, \; s_2 \mapsto -s_1, s_3 \mapsto -s_3, I_2 \mapsto I_2,$$

$\tilde{V}$ is stable under the $\mathbb{Z}/2\mathbb{Z}$-action. The element s_1 in $\tilde{V}$ gives an example of an element of the form in (13).

Consider the graded twisting $C(\tilde{V})^{t,\alpha} = C(\tilde{V})_{\text{even}} \oplus (C(\tilde{V})_{\text{odd}} \cdot \lambda_g)$ of the Hopf algebra $C(\tilde{V})$ of continuous functions on $\tilde{V}$.

We are now ready to state our main result.

Theorem 56. [10] There exists a surjective Hopf $*$-homomorphism from $C(SU_{-1}(2))$ onto $C(G_{\mathrm{KP}})$. It can be constructed by a composition of the Hopf $*$-isomorphism $C(SU_{-1}(2)) \to C(SU(2))^{t,\alpha}$, a surjective Hopf $*$-homomorphism $C(SU(2))^{t,\alpha} \to C(\tilde{V})^{t,\alpha}$, and the Hopf $*$-isomorphism $C(\tilde{V})^{t,\alpha} \to C(G_{\mathrm{KP}})$ defined by

$$(\epsilon, \alpha', \beta', \gamma') \mapsto (\epsilon, \gamma, \alpha, \beta),$$
$$M_2(\mathbb{C}) \ni x \mapsto vxv^* \qquad \left(v = \begin{pmatrix} -1 & 0 \\ 0 & i \end{pmatrix} \right).$$

There is a surjective homomorphism $C(SU(2))^{t,\alpha} \to C(\tilde{V})^{t,\alpha}$, it represents a quantum subgroup of $SU_{-1}(2)$.

Proposition 57. The Hopf $*$-algebra $C(\tilde{V})^{t,\alpha}$ is noncommutative and non-cocommutative.

Proof. Indeed, if we take a product of an element $\delta_{s_1} + \delta_{-s_1}$ in $C(\tilde{V})_{\mathrm{even}}$ and an element $(\delta_{s_2} - \delta_{-s_2})\lambda_g$ in $C(\tilde{V})_{\mathrm{odd}} \cdot \lambda_g$ in this order, then we have

$$(\delta_{s_1} + \delta_{-s_1})(\delta_{s_2} - \delta_{-s_2})\lambda_g = 0.$$

On the other hand,

$$(\delta_{s_2} - \delta_{-s_2})\lambda_g(\delta_{s_1} + \delta_{-s_1}) = (\delta_{s_2} - \delta_{-s_2})(\delta_{-s_2} + \delta_{s_2})\lambda_g = (\delta_{s_2} - \delta_{-s_2})\lambda_g,$$

which shows noncommutativity of $C(\tilde{V})^{t,\alpha}$. Furthermore, the coproduct on $C(\tilde{V})^{t,\alpha}$ is induced by

$$\Delta(\delta_h) = \sum_{h=k_1k_2} \delta_{k_1} \otimes \delta_{k_2}$$

for $h \in \tilde{V}$ and (12). Using it we can see that the comultiplication Δ on $C(\tilde{V})^{t,\alpha}$ is noncocommutative by observing that $\Delta((\delta_{s_3} - \delta_{-s_3})\lambda_g) \neq$

$\Delta^{\mathrm{op}}((\delta_{s_3}-\delta_{-s_3})\lambda_g)$ for an element $(\delta_{s_3}-\delta_{-s_3})\lambda_g$ in $C(\tilde{V})^{t,\alpha}$. By direct computation we get

$$\begin{aligned}
&\Delta((\delta_{s_3}-\delta_{-s_3})\lambda_g)\\
&=(\Delta(\delta_{s_3})-\Delta(\delta_{-s_3}))(\lambda_g\otimes\lambda_g)\\
&=\{(\delta_{s_1}-\delta_{-s_1})\otimes(\delta_{s_2}-\delta_{-s_2})-(\delta_{s_2}-\delta_{-s_2})\otimes(\delta_{s_1}-\delta_{-s_1})\\
&\quad+(\delta_{s_3}-\delta_{-s_3})\otimes(\delta_{I_2}-\delta_{-I_2})+(\delta_{I_2}-\delta_{-I_2})\otimes(\delta_{s_1}-\delta_{-s_1})\}(\lambda_g\otimes\lambda_g),
\end{aligned}$$

and

$$\begin{aligned}
&\Delta^{\mathrm{op}}((\delta_{s_3}-\delta_{-s_3})\lambda_g)\\
&=\{(\delta_{s_2}-\delta_{-s_2})\otimes(\delta_{s_1}-\delta_{-s_1})-(\delta_{s_1}-\delta_{-s_1})\otimes(\delta_{s_2}-\delta_{-s_2})\\
&\quad+(\delta_{s_3}-\delta_{-s_3})\otimes(\delta_{I_2}-\delta_{-I_2})+(\delta_{I_2}-\delta_{-I_2})\otimes(\delta_{s_1}-\delta_{-s_1})\}(\lambda_g\otimes\lambda_g).
\end{aligned}$$

This concludes the proof. □

Proof of Theorem 56. The only thing we need to describe is a concrete isomorphism of Hopf algebras $(C(\tilde{V})^{\mathrm{t},\alpha},\Delta_{\mathrm{gr}})\to(C(G_{\mathrm{KP}}),\Delta_{G_{\mathrm{KP}}})$. We set generators in $C(\tilde{V})^{t,\alpha}=C(\tilde{V})_{\mathrm{even}}\otimes(C(\tilde{V})_{\mathrm{odd}}\cdot\lambda_g)$ regarded as elements in

$$\mathbb{C}\cdot\epsilon\oplus\mathbb{C}\cdot\alpha'\oplus\mathbb{C}\cdot\beta'\oplus\mathbb{C}\cdot\gamma'\oplus M_2(\mathbb{C})$$

for projections ϵ,α',β', and γ' such that the triplet (α',β',γ') is obtained from permutation of the triplet (α,β,γ) by the following mapping

$$\delta_{s_1}+\delta_{-s_1}\mapsto\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad\delta_{s_2}+\delta_{-s_2}\mapsto\begin{pmatrix}0&0\\0&1\end{pmatrix},$$
$$\delta_{s_3}+\delta_{-s_3}\mapsto\beta'+\gamma',\quad\delta_{I_2}+\delta_{-I_2}\mapsto\epsilon'+\alpha',$$
$$(\delta_{s_1}-\delta_{-s_1})\lambda_g\mapsto\begin{pmatrix}0&-1\\0&0\end{pmatrix},\quad(\delta_{s_2}-\delta_{-s_2})\lambda_g\mapsto\begin{pmatrix}0&0\\1&0\end{pmatrix},$$
$$(\delta_{s_3}-\delta_{-s_3})\lambda_g\mapsto i(\beta'-\gamma'),\quad(\delta_{I_2}-\delta_{-I_2})\lambda_g\mapsto\epsilon'-\alpha'.$$

Applying this mapping, we can see that formulas for images of elements in $C(\tilde{V})^{t,\alpha}$ by the comultiplication Δ_{gr} is given by

$$\begin{aligned}
\Delta_{\mathrm{gr}}(\epsilon) &= \epsilon\otimes\epsilon+\alpha'\otimes\alpha'+\beta'\otimes\beta'+\gamma'\otimes\gamma' \\
&\quad+\frac{1}{2}(\epsilon_{11}\otimes\epsilon_{11}-\epsilon_{12}\otimes\epsilon_{12}-\epsilon_{21}\otimes\epsilon_{21}+\epsilon_{22}\otimes\epsilon_{22}), \\
\Delta_{\mathrm{gr}}(\alpha') &= \epsilon\otimes\alpha'+\alpha'\otimes\epsilon+\beta'\otimes\gamma'+\gamma'\otimes\beta' \\
&\quad+\frac{1}{2}(\epsilon_{11}\otimes\epsilon_{11}+\epsilon_{12}\otimes\epsilon_{12}+\epsilon_{21}\otimes\epsilon_{21}+\epsilon_{22}\otimes\epsilon_{22}), \\
\Delta_{\mathrm{gr}}(\beta') &= \epsilon\otimes\beta'+\beta'\otimes\epsilon+\alpha'\otimes\gamma'+\gamma'\otimes\alpha' \\
&\quad+\frac{1}{2}(\epsilon_{11}\otimes\epsilon_{22}+i\epsilon_{12}\otimes\epsilon_{21}-i\epsilon_{21}\otimes\epsilon_{12}+\epsilon_{22}\otimes\epsilon_{11}), \\
\Delta_{\mathrm{gr}}(\gamma') &= \epsilon\otimes\gamma'+\gamma'\otimes\epsilon+\alpha'\otimes\beta'+\beta'\otimes\alpha' \\
&\quad+\frac{1}{2}(\epsilon_{11}\otimes\epsilon_{22}-i\epsilon_{12}\otimes\epsilon_{21}+i\epsilon_{21}\otimes\epsilon_{12}+\epsilon_{22}\otimes\epsilon_{11}), \\
\Delta_{\mathrm{gr}}(x) &= \epsilon\otimes x+\alpha'\otimes w_{\alpha'}xw_{\alpha'}^*+\beta'\otimes w_{\beta'}xw_{\beta'}^*+\gamma'\otimes w_{\gamma'}xw_{\gamma'}^* \\
&\quad+x\otimes\epsilon+\overline{w}_{\alpha'}x\overline{w}_{\alpha'}^*\otimes\alpha'+\overline{w}_{\beta'}x\overline{w}_{\beta'}^*\otimes\beta'+\overline{w}_{\gamma'}x\overline{w}_{\gamma'}^*\otimes\gamma',
\end{aligned} \tag{14}$$

for projections $\epsilon,\alpha',\beta',\gamma'$ and $x\in M_2(\mathbb{C})$, where ϵ_{ij} are the matrix units and

$$w_{\alpha'}=\begin{pmatrix}-1&0\\0&1\end{pmatrix},\quad w_{\beta'}=\begin{pmatrix}0&1\\i&0\end{pmatrix},\quad w_{\gamma'}=\begin{pmatrix}0&-i\\-1&0\end{pmatrix}.$$

We can observe that these unitary matrices $w_{\alpha'}$, $w_{\beta'}$ and $w_{\gamma'}$ are transformed to unitary matrices u_γ, u_α and u_β in the formula of $\Delta_{G_{\mathrm{KP}}}$ respectively, by taking adjoint by a unitary matrix

$$v=\begin{pmatrix}-1&0\\0&i\end{pmatrix},\quad vw_{\alpha'}v^*=u_\gamma, vw_{\beta'}v^*=u_\alpha, vw_{\gamma'}v^*=u_\beta. \tag{15}$$

Moreover, $\Delta_{\mathrm{gr}}(\epsilon)$, $\Delta_{\mathrm{gr}}(\alpha')$, $\Delta_{\mathrm{gr}}(\beta')$ and $\Delta_{\mathrm{gr}}(\gamma')$ coincide with $\Delta_{G_{\mathrm{KP}}}(\epsilon)$, $\Delta_{G_{\mathrm{KP}}}(\gamma)$, $\Delta_{G_{\mathrm{KP}}}(\alpha)$ and $\Delta_{G_{\mathrm{KP}}}(\beta)$ respectively, by Φ defined by

$$(\epsilon,\alpha',\beta',\gamma')\mapsto(\epsilon,\gamma,\alpha,\beta),\ M_2(\mathbb{C})\ni x\mapsto vxv^*.$$

Then (15) implies that Φ intertwines (14) to (10). For instance, $\alpha' \otimes w_{\alpha'} x w_{\alpha'}^*$ in (14) is transformed to $\gamma \otimes v w_{\alpha'} v^* (v x v^*) v w_{\alpha'}^* v^* = \gamma \otimes u_\gamma \mathrm{Ad}_v(x) u_\gamma^*$ in (10). Hence we obtain the isomorphism $(C(\tilde{V})^{\mathrm{t},\alpha}, \Delta_{\mathrm{gr}}) \to (C(G_{\mathrm{KP}}), \Delta_{G_{\mathrm{KP}}})$. $\square$

7 $\operatorname{Rep} SU_{-1}(2)$-module homomorphisms

7.1 Representations of G_{KP}

The 1-dimensional representations of G_{KP} are following.

$$
\begin{aligned}
u_1 &= \epsilon + \alpha + \beta + \gamma + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
u_2 &= \epsilon - \alpha - \beta + \gamma + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
u_3 &= \epsilon + \alpha + \beta + \gamma + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \\
u_4 &= \epsilon - \alpha - \beta + \gamma + \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}
\tag{16}
$$

The oriented graph with weights corresponding to the representation category $\operatorname{Rep}(G_{\mathrm{KP}})$ is in Figure 1. Each vertex corresponds to an irreducible object in $\operatorname{Rep}(G_{\mathrm{KP}})$ with labeling corresponding to the convention of Section 2.2. Total weights on the oriented edges starting from one vertex is equal to 2. See [4] for the interpretation of the weights of this graph.

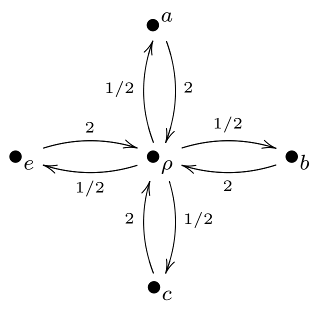


Figure 1: $D_4^{(1)}$

Let $U \in M_2(C(G_{\mathrm{KP}}))$ be the fundamental representation u_i of G_{KP} (16). Its tensor product $U \otimes U$ decomposes into $\sum_{i=1}^{4} P_i \otimes u_i$ wth mutually orthogonal matrices

$$P_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \frac{1}{4}\begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix},$$

$$P_3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}, \quad P_4 = \frac{1}{4}\begin{pmatrix} 1 & -1 & 1 & 1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 \end{pmatrix}.$$

7.2 Concrete computation

Unitary maps ψ consisting a $\operatorname{Rep} SU_{-1}(2)$-module homomorphism (G, ψ) together with a functor $G\colon \operatorname{Rep} G_{\mathrm{KP}} \to \mathrm{Hilb}_f$ can be given by solving equations provided by the interpretation of conditions on the natural equivalence ψ in terms of bigraded vector spaces [3, 4].

From the information in the graph in Figure 1 we can write up the q-fundamental solution in $\operatorname{Rep} G_{\mathrm{KP}}$. Recall that $\mathrm{Irr} G_{\mathrm{KP}} = K_4 \cup \{\rho\}$. The bigraded vector spaces associated with the $\operatorname{Rep} SU_{-1}(2)$-module category $\operatorname{Rep} G_{\mathrm{KP}}$ are denoted by $H_{\rho g}$ and $H_{g\rho}$ for $g \in K_4 = \{e, a, b, c\}$. They are all one dimensional so we write unit vectors as $\xi_{\rho g} \in H_{\rho g}$ and $\xi_{g\rho} \in H_{g\rho}$. Therefore the q-fundamental solution R in $\operatorname{Rep} G_{\mathrm{KP}}$ is described by the vectors

$$\sqrt{2}\xi_{g\rho} \otimes \xi_{\rho g},$$

and

$$\frac{1}{\sqrt{2}} \sum_{g \in K_4} \xi_{\rho g} \otimes \xi_{g\rho}.$$

The vector spaces associated with the the $\operatorname{Rep} SU_{-1}(2)$-module category Hilb_f are H_ρ and H_g of dimensions 2 and 1. Here the unital maps of the

$\operatorname{Rep} SU_{-1}(2)$-module homomorphisms are expressed as

$$\psi_g \colon H_\rho \otimes H_{\rho g} \to H_{1/2} \otimes H_g$$

for $g \in K_4 = \{e, a, b, c\}$ and

$$\psi_\rho \colon \bigoplus_{g \in K_4} H_g \otimes H_{g\rho} \to H_{1/2} \otimes H_\rho$$

satisfying the following commutative diagrams:

$$\begin{array}{ccc} H_g & \xrightarrow{\mathrm{id}\otimes \mathrm{R}} & H_g \otimes H_{g\rho} \otimes H_{\rho g} \\ & \searrow^{\mathrm{R}\otimes \mathrm{id}} & \downarrow{\scriptstyle (\mathrm{id}\otimes\psi_g)(\psi_\rho\otimes \mathrm{id})} \\ & & H_{1/2} \otimes H_{1/2} \otimes H_g, \end{array} \qquad \begin{array}{ccc} H_\rho & \xrightarrow{\mathrm{id}\otimes \mathrm{R}} & \bigoplus_{g\in K_4} H_\rho \otimes H_{\rho g} \otimes H_{g\rho} \\ & \searrow^{\mathrm{R}\otimes \mathrm{id}} & \downarrow{\scriptstyle (\mathrm{id}\otimes\psi_\rho)(\psi_g\otimes \mathrm{id})} \\ & & H_{1/2} \otimes H_{1/2} \otimes H_\rho. \end{array}$$

From the projections $P_i \in M_4(\mathbb{C})$ in the tensor product of fundamental representation U of G_{KP} with itself $U \otimes U = \sum_{i=1}^4 p_i \otimes u_i$, we can compute the maps ψ_g, ψ_ρ concretely.

Let $\{\xi_1, \xi_2\}$ be an orthonormal basis of H_ρ.

Theorem 58. The unitary maps

$$\psi_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \psi_a = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \psi_b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \psi_c = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

$$\psi_\rho = \frac{1}{2} \begin{pmatrix} 0 & 1 & \sqrt{2} & 1 \\ \sqrt{2} & 1 & 0 & -1 \\ \sqrt{2} & -1 & 0 & 1 \\ 0 & 1 & \sqrt{2} & 1 \end{pmatrix}$$

associated with the $\operatorname{Rep} SU_{-1}(2)$-module homomorphism $\operatorname{Rep} G_{\mathrm{KP}} \to \mathrm{Hilb}_{\mathrm{f}}$ make the above diagrams commutes. Here the matrix presentation of ψ_ρ is with respect to the basis

$$\{\xi_e \otimes \xi_{e\rho},\ \xi_a \otimes \xi_{a\rho},\ \xi_b \otimes \xi_{b\rho},\ \xi_c \otimes \xi_{c\rho}\}$$

of $\bigoplus_{g\in K_4} H_g \otimes H_{g\rho}$ and the basis

$$\{e_1 \otimes \xi_1,\ e_1 \otimes \xi_2,\ e_2 \otimes \xi_1,\ e_2 \otimes \xi_2\}$$

of $H_{1/2} \otimes H_\rho$.

Proof. We show that the equation

$$(\mathrm{id} \otimes \psi_g)(\psi_\rho \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{R})(\xi_g) = (\mathrm{R} \otimes \mathrm{id})(\xi_g) \tag{17}$$

holds for the unit vector ξ_g in H_g in the case $g = b$. On the right hand side we have

$$(\mathrm{R} \otimes \mathrm{id})(\xi_b) = (e_1 \otimes e_1 + e_2 \otimes e_2) \otimes \xi_b,$$

while on the left hand side we have

$$\begin{aligned}(\mathrm{id} \otimes \psi_b)(\psi_\rho \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{R})(\xi_b) &= \sqrt{2}(\mathrm{id} \otimes \psi_b)(\psi_\rho \otimes \mathrm{id})(\xi_b \otimes \xi_{b\rho} \otimes \xi_{\rho b}) \\ &= \sqrt{2}(\mathrm{id} \otimes \psi_b)\left(\frac{1}{\sqrt{2}}(e_1 \otimes \xi_1 - e_2 \otimes \xi_2)\right) \\ &= e_1 \otimes e_1 \otimes \xi_b + e_2 \otimes e_2 \otimes \xi_b.\end{aligned}$$

Therefore, (17) for $g = b$ holds. Other cases can be shown similarly. □

8 Three-state asymmetric simple exclusion processes

8.1 Asymmetric simple exclusion processes

An asymmetric simple exclusion process, called ASEP for short, is an exclusion process on N site one-dimensional lattice, which describes one-dimensional random walks. One particle is admitted in each site. There is no out-going particles and in-coming particles. The particles moves left and right and its rule is determined by the local transition rates. We denote by p_R for the rate of particle moving right and p_L for left.

We express the system using the vectors

$$|0\rangle_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle_i = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

to describe each site contain a particle or not. Then the whole system is expressed as

$$|\tau_1, \ldots, \tau_N\rangle = |\tau_1\rangle_1 \otimes \cdots \otimes |\tau_N\rangle_N$$

with $\tau_i \in \{0, 1\}$. The probabilities to get each configuration are denoted by $p(t; \tau_1, \ldots, \tau_N)$. Then the configuration at time t is the formula

$$|P(t)\rangle = \sum_{\tau_i \in \{0,1\}} p(t; \tau_1, \ldots, \tau_N) |\tau_1, \ldots, \tau_N\rangle .$$

The time evolution of $|P(t)\rangle$ is given by the differential equation

$$\begin{aligned} \frac{d}{dt} p(t; \tau_1, \ldots, \tau_N) = & \sum_{i=1}^{N} \Theta(\tau_{i+1} - \tau_i) p(t; \tau_1, \ldots, \tau_{i+1}, \tau_i, \ldots, \tau_N) \\ & - \sum_{i=1}^{N} \Theta(\tau_i - \tau_{i+1}) p(t; \tau_1, \ldots, \tau_i, \tau_{i+1}, \ldots, \tau_N), \end{aligned} \tag{18}$$

where Θ is determined by

$$\Theta(x) = \begin{cases} -p_R & (x < 0) \\ 0 & (x = 0) \\ p_L & (x > 0). \end{cases}$$

The equation (18) in a matrix form is given by

$$\frac{d}{dt} |P(t)\rangle = M |P(t)\rangle ,$$

where $M = \sum_{i=1}^{N-1} M_{i,i+1}$ is the Markov matrix determined by

$$M_{i.i+1} = \mathbb{1}_1 \otimes \cdots \otimes \mathbb{1}_{i-1} \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -p_L & p_R & 0 \\ 0 & p_L & -p_R & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1} \otimes \mathbb{1}_{i+2} \otimes \cdots \otimes \mathbb{1}_N .$$

The matrix $M_{i.i+1}$ contributes only to i and $(i+1)$-th terms among N times tensor poroducts.

Let us illustrate an example of ASEP. We consider the case where $N = 3$ and the particle transition rates given by $p_L = 0.2$ and $p_R = 0.3$. We take $p(0; 1, 0, 0) = 1$, that means the initial state is $|1, 0, 0\rangle$. Then the probability of each configuration through the time evolution starting from $t = 0$ to $t = 8$ is shown in Figure 2. The state changes as the particle is likely to move right.

8.2 $U_q(sl_2)$ and Temperley-Lieb algebra

Next we see how to identify the update operator of ASEP by the generators of Temperley-Lieb algebra.

Definition 59. For a real number q such that $0 < q \leq 1$, the algebra $U_q(sl_2)$ is generated by the elements E, FK, K^{-1} satisfying the relations

$$KK^{-1} = K^{-1}K = 1, \quad KE = q^2EK, \quad KF = q^{-2}FK,$$
$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}.$$

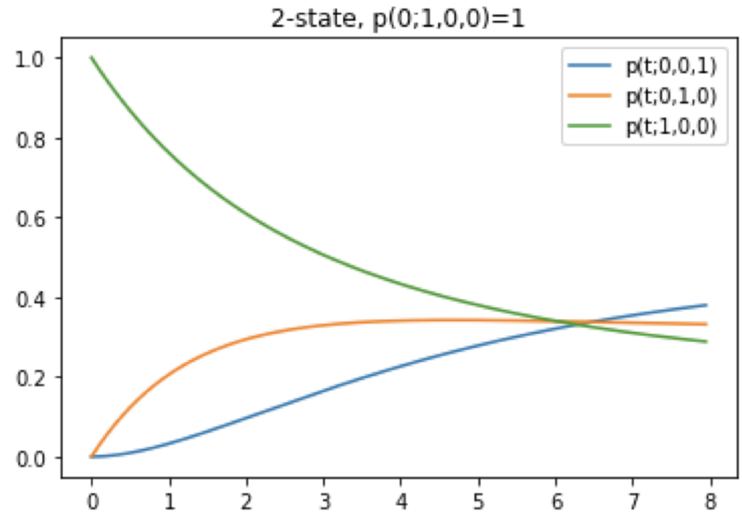


Figure 2: Changes of probability of each configuration in a ASEP

Definition 60. Temperley-Lieb algebra is the algebra generated by the elements $1, e_1, \ldots, e_{n-1}$ satisfying the relations

$$\begin{aligned} e_i^2 &= (q + q^{-1})e_i, \\ e_i e_{i+1} e_i &= e_i, \\ e_i e_j &= e_j e_i \quad (|i - j| \geq 2), \end{aligned} \tag{19}$$

for every $i = 1, \ldots, n - 1$.

We can check that the Temperley-Lieb (TL) generators e_i of the form

$$e_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

satisfy the above relations.

For the concrete computation, the use of the graphical representation of TL generators is effective. Graphs can be created by combining the identity operator $\mathbb{1}_i$ and the TL generator e_i, displayed in Figure 3. Arranging each of

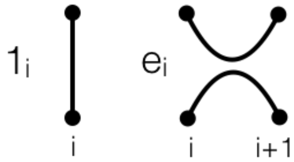


Figure 3: Graphical representation of the identity operator $\mathbb{1}_i$ and the TL generator e_i.

operators in Figure 3 vertically means the composition of the operators. On the other hand, arranging them horizontally indicates taking tensor products. Moreover, if a circle appears, then the weight $(q + q^{-1})$ is applied. Then, the TL relations (19) are drawn in Figure 4, 5, and 6.

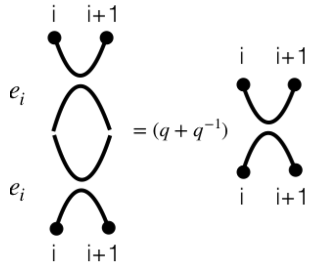


Figure 4: Graphical representation of the equation $e_i^2 = (q + q^{-1})e_i$.

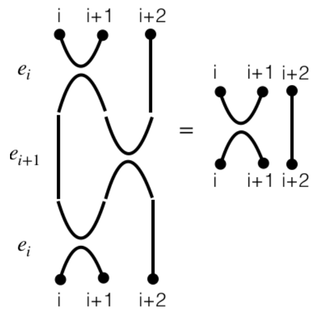


Figure 5: Graphical representation of the equation $e_i e_{i+1} e_i = e_i$.

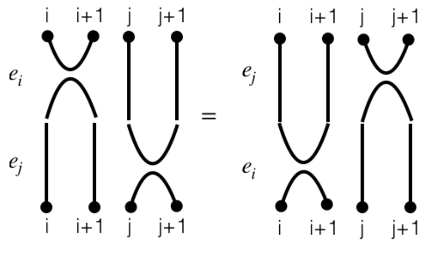


Figure 6: Graphical representation of the equation $e_i e_j = e_j e_i \quad (|i - j| \geq 2)$.

An important fact is that there is a relationship between ASEP update operators and TL generators. Define the similarity transformation matrix U

by

$$U = \bigotimes_{i=1}^{N} U_i = \bigotimes_{i=1}^{N} \begin{pmatrix} 1 & 0 \\ 0 & q^{i-1} \end{pmatrix}_i ,$$

with $q = \sqrt{p_R/p_L} > 0$. Then the update operator TL generators by the equation

$$M_{i,i+1} = -\sqrt{p_R p_L} U_{i,i+1} e_i (U_{i,i+1})^{-1}.$$

Therefore, the update operators satisfy the TL relations (19).

It is known that the spin operators defined by

$$S^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad S^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad q^S = \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix} \tag{20}$$

generates the algebra $U_q(sl_2)$. The coproduct Δ on $U_q(sl_2)$ satisfies the coassociativity

$$(\Delta \otimes \mathrm{id})\Delta(X) = (\mathrm{id} \otimes \Delta)\Delta(X)$$

for $X \in \{S^\pm, q^S\}$. Then we have

$$[e_i, \Delta(X)] = 0.$$

Therefore, the update operator $M_{i,i+1}$ of ASEP also commutes with the spin operators up to the similarity transformation.

8.3 Extension to three-state ASEP

A muliti-state extension of ASEP is developed using the higher-dimensional representation of the $U_q(sl_2)$ algebra by Matsui [12]. In this section we consider the three-state case. Our target is to construct the update operators of three-state ASEP with TL generators.

For three-state ASEP, two particles are admitted in one box. The vectors for zero, one, and two local states are following.

$$|0\rangle \otimes |0\rangle, \quad \frac{q^{\frac{1}{2}} |0\rangle \otimes |1\rangle + q^{-\frac{1}{2}} |1\rangle \otimes |0\rangle}{\| q^{\frac{1}{2}} |0\rangle \otimes |1\rangle + q^{-\frac{1}{2}} |1\rangle \otimes |0\rangle \|}, \quad |1\rangle \otimes |1\rangle$$

Let us describe three-dimensional TL generators construction. We introduce the projection operator $Y^{(2)}$ by the reccurrence formula

$$
\begin{aligned}
Y^{(2)}(e_i) &= Y^{(1)}(1 - \frac{U_0(\tau)}{U_1(\tau)} e_i) Y^{(1)} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{q}{q+q^{-1}} & \frac{1}{q+q^{-1}} & 0 \\ 0 & \frac{1}{q+q^{-1}} & \frac{q^{-1}}{q+q^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{i,i+1},
\end{aligned}
$$

where $Y^{(1)} = 1$ and $U_k(\tau)$ is the Chebyshev polynomials of the second kind with the parameter $\tau = (q + q^{-1})/2$. Then the three-dimensional fused TL generators $e_i^{(2:r)}$ are given by

$$e_i^{(2:1)} = Y^{(2)} e_{2(i-1)+2} Y^{(2)}, \tag{21}$$

$$e_i^{(2:2)} = Y^{(2)} e_{2(i-1)+2} e_{2(i-1)+1} e_{2(i-1)+3} e_{2(i-1)+2} Y^{(2)}. \tag{22}$$

Similar to the usual TL generators e_i, for the spin operators $X \in \{S^{\pm}, q^S\}$ introduced in (20) we have that

$$[e_i^{(2:r)}, \Delta(X)] = 0.$$

Figure 7 and 8 the graphs for the three-dimensional fused TL generators $e_i^{2:r}$. The rings express the projection operator $Y^(2)$.

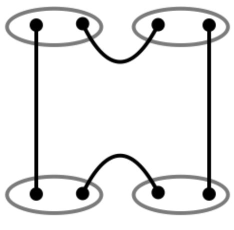


Figure 7: Graphical representation of the equation $e_i^2 = (q + q^{-1}) e_i$.

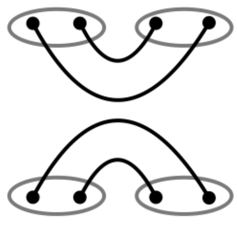


Figure 8: Graphical representation of the equation $e_i e_{i+1} e_i = e_i$.

The conditions to be the update operator of three-state ASEP is as follows.

(i) (the principle of probability conservation) The sum of each column should be zero.

(ii) (the positivity of probability) The diagonal element should be negative values, while the off-diagonal elements should be positive values.

The matrix of three-dimensional fused TL generators $e_i^{(2:1)}$ of type 1 created by the formula (21) is given by the matrix below.

$$e_i^{(2;1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q^{-2}}{q+q^{-1}} & 0 & -\frac{1}{q+q^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 & -\frac{q}{(q+q^{-1})^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{q+q^{-1}} & 0 & \frac{q^2}{q+q^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -q & 0 & \frac{q^2-1+q^{-2}}{q+q^{-1}} & 0 & -\frac{1}{q} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q^{-2}}{q+q^{-1}} & 0 & -\frac{1}{q+q^{-1}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{q^{-1}}{(q+q^{-1})^2} & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{q+q^{-1}} & 0 & \frac{q^2}{q+q^{-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

Clearly, by multiplying -1 to $e_i^{(2:1)}$, it becomes to satisfy the condition (ii).

In order to deal with the condition (i), we need the similarity transformation $U^{(2)}$ of three-state ASEP given by

$$U^{(2)} = \bigotimes_{i=1}^{N} U_{i,i+1}^{(2)} = \bigotimes_{i=1}^{N} \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{2(i-1)} & 0 \\ 0 & 0 & q^{4(i-1)} \end{pmatrix}_{i,i+1} .$$

Then the sum of each column in the matrix $U_{i,i+1}e_i^{(2;1)}(U_{i,i+1})^{-1}$ becomes zero. Indeed, the matrix is computes as

$$U^{(2)}_{i,i+1} e_i^{(2;1)} (U^{(2)}_{i,i+1})^{-1}$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{q^{-2}}{q+q^{-1}} & 0 & -\frac{q^2}{q+q^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{q} & 0 & -\frac{q^3}{(q+q^{-1})^2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & \frac{q^2}{q+q^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{q} & 0 & \frac{q^2-1+q^{-2}}{q+q^{-1}} & 0 & -q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{q^{-2}}{q+q^{-1}} & 0 & -\frac{q^2}{q+q^{-1}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{q^{-3}}{(q+q^{-1})^2} & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & \frac{q^2}{q+q^{-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1} .$$

Similarly, applying the same similarity transformation $U^{(2)}$ to the three-dimensional fused TL generator $e_i^{(2:2)}$ of type determined by (22), it satisfies the condition (ii).

Now we are ready to state the following proposition.

Proposition 61. [12] Take the matrices

$$M^{(2;r)}_{i,i+1} = -U_{i,i+1} e_i^{(2;r)} (U_{i,i+1})^{-1} \tag{23}$$

for $r = 1, 2$. Then the linear combination

$$M^{(2)}_{i,i+1} = b_1 M^{(2;1)}_{i,i+1} + b_2 M^{(2;2)}_{i,i+1} \tag{24}$$

satisfies the positivity condition as long as $\beta = b_2/b_1$ satisfies

$$\begin{cases} -\frac{q^2}{q+q^{-1}} < \beta < 0 & (0 < q \leq 1), \\ -\frac{q^{-2}}{q+q^{-1}} < \beta < 0 & (1 \leq q). \end{cases}$$

By the formula (24) we get the update operator as

$$M^{(2)}_{i,i+1} =$$

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{q^2(q+q^{-1})} & 0 & \frac{q^2}{q+q^{-1}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{q+\beta}{q^2} & 0 & -\frac{q^3+\beta q(q+q^{-1})}{(q+q^{-1})^2} & 0 & -q^4\beta & 0 & 0 \\
0 & -\frac{1}{q^2(q+q^{-1})} & 0 & -\frac{q^2}{q+q^{-1}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\frac{q^3+\beta q(q+q^{-1})}{q^4} & 0 & -\frac{q^4-q^2+1+\beta q^2(q+q^{-1})}{q^2(q+q^{-1})} & 0 & -q(\beta q^3+\beta q+1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{q^2(q+q^{-1})} & 0 & \frac{q^2}{q+q^{-1}} & 0 \\
0 & 0 & 0 & 0 & -\frac{\beta q^2(q+q^{-1})+1}{q^3(q+q^{-1})^2} & 0 & -q(q\beta+1) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{q^{-2}}{q+q^{-1}} & 0 & \frac{q^2}{q+q^{-1}} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}_{i,i+1}$$

with $\beta = b_2/b_1$. Using this matrix, the direct computation provides the proof of the proposition.

Let us see an example of three-state ASEP. Take $N = 3$ of the length of the lattice and $p(t; 1, 0, 1) = 1$ as the initial state of $|1, 0, 1\rangle$. The local transition rates are given by $p_L = 0.2$ and $p_R = 0.3$. Then the probability of each configuration from time $t = 0$ to $t = 8$ is described in Figure 9.

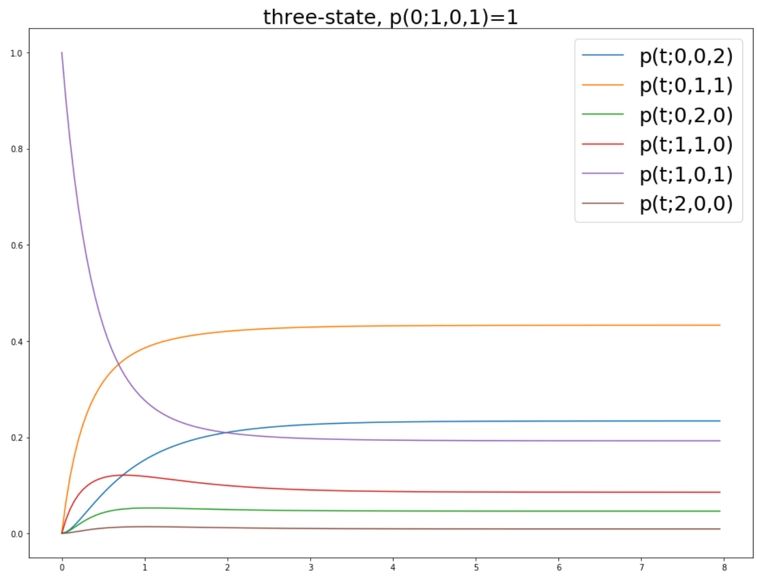


Figure 9: Changes of probability of each configuration in a three-state ASEP

8.4 Investigation of the case where q is negative

Now we are interested in the investigation of the case where q is negative. We begin with the setting of the ingredients with $q < 0$. In this case, the TL generators e_i are given by the matrices (25). The sign is switched and q is replaced by the absolute values of q.

$$e_i = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & |q|^{-1} & 1 & 0 \\ 0 & 1 & |q| & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1} \tag{25}$$

Similarly, we change the sign and absolute values for the vectors for zero, one, and two particles.

$$|0\rangle \otimes |0\rangle\,, \quad \frac{|q|^{\frac{1}{2}}\,|0\rangle \otimes |1\rangle - |q|^{-\frac{1}{2}}\,|1\rangle \otimes |0\rangle}{\| |q|^{\frac{1}{2}}\,|0\rangle \otimes |1\rangle - |q|^{-\frac{1}{2}}\,|1\rangle \otimes |0\rangle \,\|}, \quad |1\rangle \otimes |1\rangle$$

Moreover, the projection operator $Y^{(2)}$ is given by the formula (26).

$$Y^{(2)}(e_i) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{|q|}{|q|+|q|^{-1}} & \frac{1}{|q|+|q|^{-1}} & 0 \\ 0 & \frac{1}{|q|+|q|^{-1}} & \frac{q^{-1}}{|q|+|q|^{-1}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{i,i+1}, \tag{26}$$

Then we can compute the matrices for the three-dimensional fused TL generators $e_i^{(2:r)}$ given by (21) and (22) as follows.

$$e_i^{(2;1)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{|q|^{-1}}{|q|^2(|q|+|q|^{-1})} & 0 & \frac{-1}{|q|+|q|^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{|q|} & 0 & -\frac{|q|}{|q|+|q|^{-1}} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{|q|+|q|^{-1}} & 0 & \frac{|q|^2}{|q|+|q|^{-1}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{|q|}{|q|+|q|^{-1}} & 0 & -\frac{|q|^{-2}-1+|q|^2}{|q|+|q|^{-1}} & 0 & -\frac{|q|}{|q|+|q|^{-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{|q|^{-2}}{|q|+|q|^{-1}} & 0 & -\frac{1}{|q|+|q|^{-1}} & 0 \\ 0 & 0 & 0 & 0 & -\frac{|q|^{-1}}{|q|+|q|^{-1}} & 0 & q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{|q|+|q|^{-1}} & 0 & \frac{|q|^2}{|q|+|q|^{-1}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

$$e_i^{(2:2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{|q|^2} & 0 & \frac{|q|^3-2|q|+|q|^{-1}}{|q|+|q|^{-1}} & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{|q|-2|q|^{-1}+|q|^{-3}}{(|q|+|q|^{-1})^2} & 0 & \frac{(|q|^2-2|q|^{-1}+|q|^{-2})^2}{(|q|+|q|^{-1})^4} & 0 & \frac{|q|^3-2|q|+|q|^{-1}}{|q|+|q|^{-1}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{|q|^3-2|q|+|q|^{-1}}{|q|+|q|^{-1}} & 0 & |q|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

In the similarity transformation matrix $U^{(2)}$, we add an entry with the negative value.

$$U^{(2)} = \bigotimes_{i=1}^{N} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -|q|^{2(i-1)} & 0 \\ 0 & 0 & |q|^{4(i-1)} \end{pmatrix}$$

Then we construct the pieces $M^{(2:r)}$ of the update operator by the same formula (23) as before. For example, we obtain the matrix of $M_{i,i+1}^{(2:2)}$ as below.

$$M_{i,i+1}^{(2:2)} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{|q|^2} & 0 & -\frac{|q|(1-2|q|^2+|q|^4)}{(|q|^{-1}+|q|)(1+|q|^2)^2} & 0 & |q|^4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{(|q|^{-1}+|q|)(1-2|q|^2+|q|^4)}{|q|^3(1+|q|^2)^2} & 0 & \frac{(1-2|q|^2+|q|^4)^2}{(1+|q|^2)^4} & 0 & -\frac{|q|^3(|q|^{-1}+|q|)(1-2|q|^2+|q|^4)}{(1+|q|^2)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{|q|^4} & 0 & -\frac{1-2|q|^2+|q|^4}{|q|(|q|^{-1}+|q|)(1+|q|^2)^2} & 0 & |q|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{i,i+1}$$

Consider a matrix W defined by the following formula.

$$W = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{(1-|q|)^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Using this matrix W, we take WXW for $X = M^{(2:r)}_{i,i+1}$, that is X multiplied by the value $1/(1-|q|)^2$ only in the fifth line and fifth column of X, and then we can see that we obtain the same matrices $M^{(2:r)}$ as in the case of positive q. As a conclusion, it provide the update operator for the same process as the case of positive q.

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